Kermack-McKendrick type models for epidemics with nonlocal aggregation terms

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Gradient Flow Face-to-Face 5, Granada



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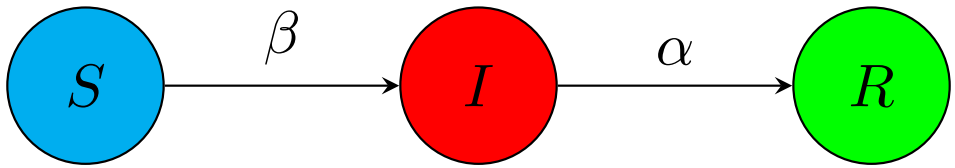
[0][Di Francesco and Ghaderi Z, 2025]

Main Model

SIR Model [Kermack and McKendrick, 1927] **with nonlocal movement**

- $x \in \mathbb{R}^2$, $t \in [0, T]$
- $S(x,t)$: The number of susceptible individuals in x at time t.
- $I(x,t)$: The number of infected individuals in x at time t.
- $R(x,t)$: The number of recovered individuals in x at time t.
- β: Infectious rate.
- α: Recovery rate.

$$\begin{cases} \partial_t S = \operatorname{div}\left(S \nabla V_S[S,I,R]\right) - \beta SI, \\ \partial_t I = \operatorname{div}\left(I \nabla V_I[S,I,R]\right) + \beta SI - \alpha I, \\ \partial_t R = \operatorname{div}\left(R \nabla V_R[S,I,R]\right) + \alpha I. \end{cases} \tag{1}$$

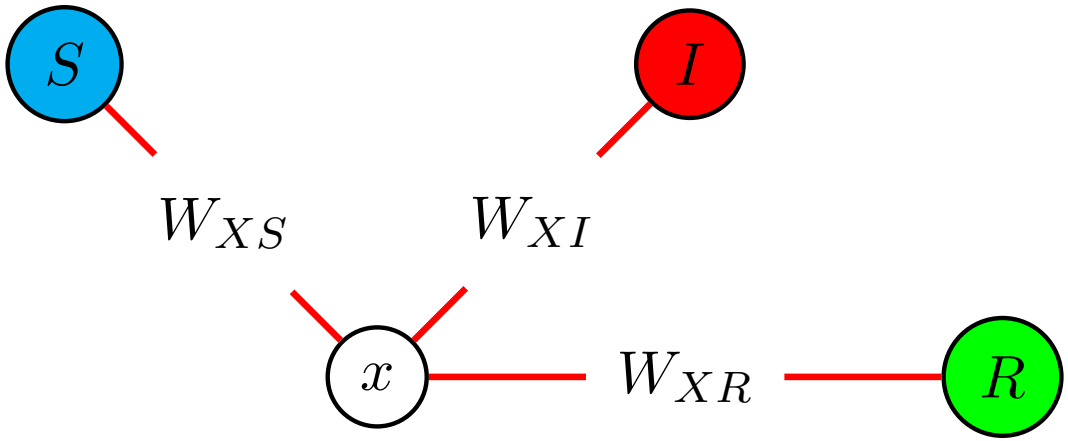


Spatial Movement

The spatial movement is mostly described using:

- Diffusion [Langlais and Busenberg, 1997, Kim, 2006]
- Nonlocal movement [Kuniya and Wang, 2018, Yang and Wang, 2023, Bentout et al., 2023, Li, 2023]

Nonlocal Social Interaction: Let x be an individual in compartment X ($X \in \{S, I, R\}$). The movement of x is affected differently by other individuals in each compartment based on their affiliation.



We assume that the kernels satisfy the following conditions.

$$D^\gamma W_{\xi\eta} \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$$
$$\text{for all multi-indexes } \gamma \text{ with } 1 \leq |\gamma| \leq 2 \text{ and for all } \xi, \eta \in \{S, I, R\}. \quad (2)$$

$$W_{\xi\eta}(-x) = W_{\xi\eta}(x) \qquad \text{for all } x \in \mathbb{R}^2 \text{ and for all } \xi, \eta \in \{S, I, R\}. \quad (3)$$

The operators V_S, V_I, and V_R model nonlocal aggregation of the individuals in each compartment and are defined as:

$$\begin{aligned} V_S[S, I, R] &= W_{SS} * S + W_{SI} * I + W_{SR} * R, \\ V_I[S, I, R] &= W_{IS} * S + W_{II} * I + W_{IR} * R, \\ V_R[S, I, R] &= W_{RS} * S + W_{RI} * I + W_{RR} * R. \end{aligned} \quad (4)$$

Main Theorem

Theorem 5 [Existence and uniqueness of solutions]

Assume the interaction kernels $W_{\xi\eta}$ satisfy (2) and (3). Assume the initial data S_0, I_0, R_0 belong to $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$ and are nonnegative. Then, for any arbitrary time $T \geq 0$, there exists one and only one

$$(S, I, R) \in L^\infty([0, T]\,;\ (L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)))^3$$

solving (1) in the weak sense and having (S_0, I_0, R_0) as initial datum.

Approximated problem

We consider, for a fixed $\varepsilon > 0$, the approximated problem

$$\begin{cases} S_t = \operatorname{div}\left(S\nabla V_S[S,I,R]\right) - \beta SI + \varepsilon\Delta S, \\ I_t = \operatorname{div}\left(I\nabla V_I[S,I,R]\right) + \beta SI - \alpha I + \varepsilon\Delta I, \\ R_t = \operatorname{div}\left(R\nabla V_R[S,I,R]\right) + \alpha I + \varepsilon\Delta R. \end{cases} \tag{5}$$

Proposition 6

Assume (2) and let the initial condition satisfy $(S_0, I_0, R_0) \in \left(L^1 \cap L^\infty\left(\mathbb{R}^2\right)\right)^3$. Then, there exists $T > 0$ such that there exists one and only one classical solution $(S, I, R) \in (C^{1,2}_{t,x}(\mathbb{R}^2 \times [0,T]))^3$ to system (5) in $\mathbb{R}^2 \times [0,T]$ with initial condition (S_0, I_0, R_0).

Proposition 7 [Uniform L^p estimates and global existence]

Let (S, I, R) be the unique local solution to the system (5) found in Proposition 6 and let $p \in [1, +\infty]$. Under the same assumptions of Proposition 6, assuming further $S_0, I_0, R_0 \in \left(L^1 \cap L^\infty\left(\mathbb{R}^2\right)\right)$ are nonnegative functions, then, there exists a constant $C \geq 0$ independent of time and of ε such that

$$\begin{aligned} &\|S(\cdot,t)\|_{L^p(\mathbb{R}^2)} + \|I(\cdot,t)\|_{L^p(\mathbb{R}^2)} + \|R(\cdot,t)\|_{L^p(\mathbb{R}^2)} \\ &\quad \leq \left(1 + \|S_0\|_{L^p(\mathbb{R}^2)} + \|I_0\|_{L^p(\mathbb{R}^2)} + \|R_0\|_{L^p(\mathbb{R}^2)}\right) e^{Ct(1+e^{Ct})}, \end{aligned} \tag{6}$$

for all $t \in [0, T]$. Consequently, the solution (S, I, R) found in Proposition 6 exists globally in time.

$\Rightarrow$ the family of solutions $(S_\varepsilon, I_\varepsilon, R_\varepsilon)$ is weakly compact in L^p

$$\begin{cases} \partial_t S = \operatorname{div}\left(S\nabla V_S[S,I,R]\right) - \beta SI, \\ \partial_t I = \operatorname{div}\left(I\nabla V_I[S,I,R]\right) + \beta SI - \alpha I, \\ \partial_t R = \operatorname{div}\left(R\nabla V_R[S,I,R]\right) + \alpha I. \end{cases}$$

Proposition 8 [Uniform H^1 estimates]

Let (S, I, R) be the unique solution to the system (5). Assume the interaction kernels satisfy (2) and that the initial functions S_0, I_0, R_0 belong to $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$ and are nonnegative. Then, for a given $T \geq 0$, the H^1-norm of $(S(\cdot,t), I(\cdot,t), R(\cdot,t))$ is uniformly bounded on $[0,T]$ with respect to $\varepsilon > 0$.

Proposition [Vanishing viscosity limit]

Let $(S_\varepsilon, I_\varepsilon, R_\varepsilon)$ be the unique solution to the system (5) with the diffusion constant $\varepsilon > 0$. Under the assumptions of Theorem 5, there exists a solution

$$(S, I, R) \in L^\infty([0,T]; H^1(\mathbb{R}^2))^3$$

that solves (1) which is the $\varepsilon \searrow 0$ limit of a subsequence of $(S_\varepsilon, I_\varepsilon, R_\varepsilon)$ in the $L^2(\mathbb{R}^2 \times [0,T])$-sense for all $T \geq 0$.

Proposition 9 [Uniqueness]

Under the same assumptions of Theorem 5, for a fixed $T \geq 0$, there exists at most one weak solution to (1) with $L^\infty([0,T]\,;\ L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \cap H^1(\mathbb{R}^2))$ regularity with a given initial condition (S_0, I_0, R_0) satisfying the assumptions of Theorem 5.

SIS model with Nonlocal movement

Consider the following SIS Model with Nonlocal movement.

$$\partial_t S = \partial_x(SW' * (S+I)) - \beta SI + \alpha I\,,$$
$$\partial_t I = \partial_x(IW' * (S+I)) + \beta SI - \alpha I\,.$$

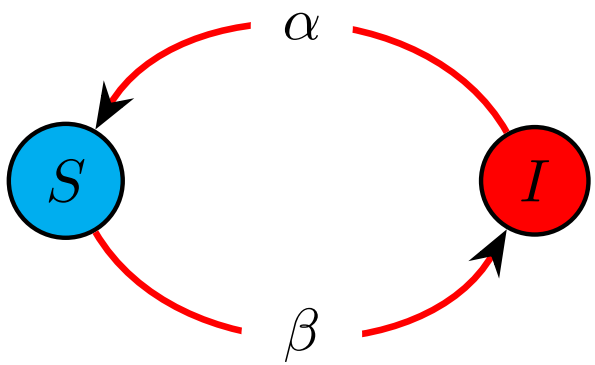
We restrict for simplicity to the one-space dimensional case and we set

$$W(x) = x^2 - \gamma|x| \tag{7}$$

for some constant $\gamma > 0$.

Basic Reproduction Number:

$$\mathcal{R}_0 = \frac{M\beta}{\gamma\alpha}$$



where $N = S + I$ and

$$M = \int_{\mathbb{R}} N(x)dx > 0\,.$$

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Theorem 11 [Steady states]

The stationary system of the model (7) with W given by (7) with $\gamma > 0$ has the one-parameter family of solutions

$$S(x) = \frac{M}{\gamma}\mathbf{1}_{[-\gamma/2,\gamma/2]}(x)\,, \qquad I(x) = 0$$

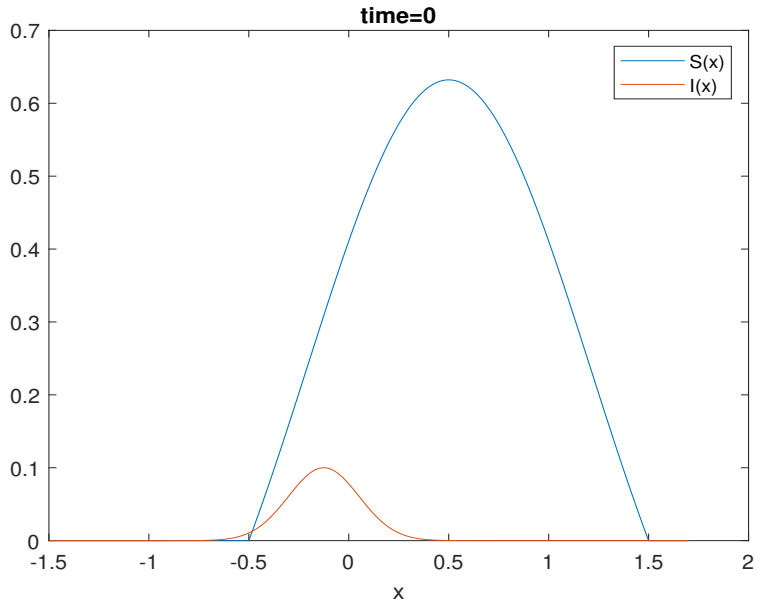
for all $M \geq 0$. Moreover, it has the additional one-parameter family of solutions

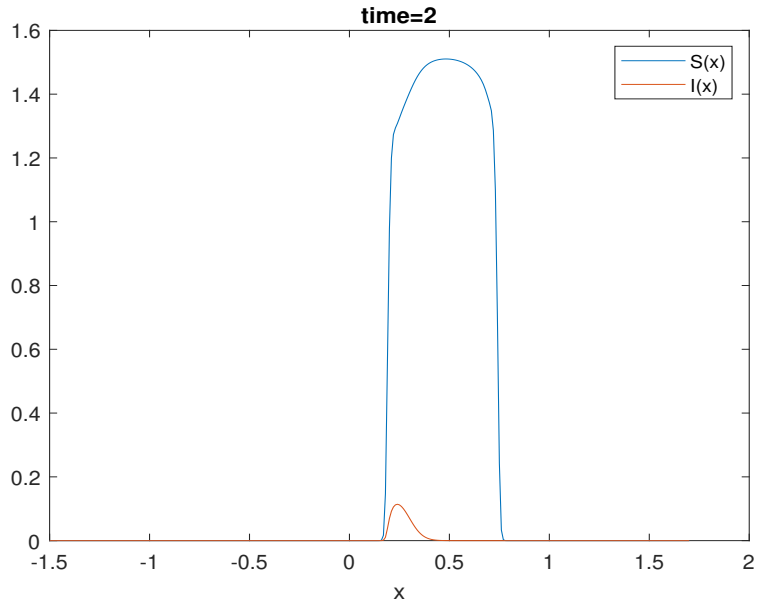
$$S(x) = \frac{\alpha}{\beta}\mathbf{1}_{[-\gamma/2,\gamma/2]}(x)\,, \qquad I(x) = \left(\frac{M}{\gamma} - \frac{\alpha}{\beta}\right)\mathbf{1}_{[-\gamma/2,\gamma/2]}(x)$$

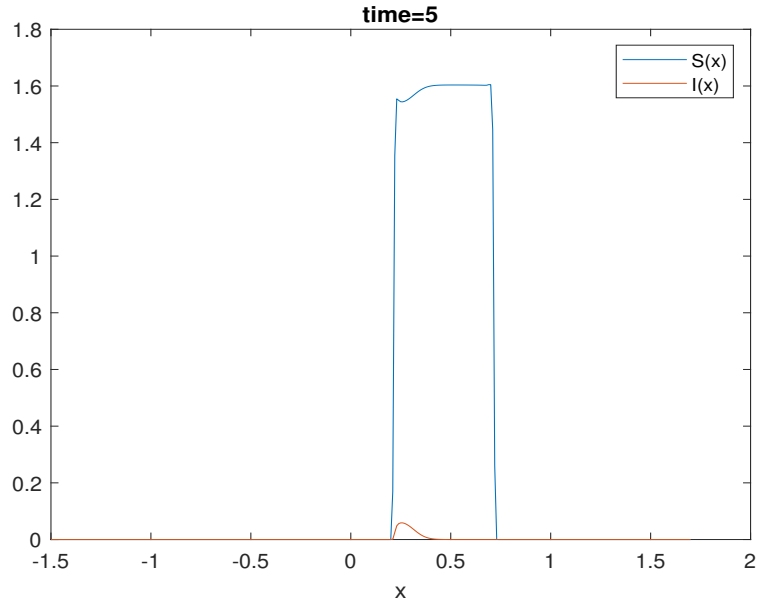
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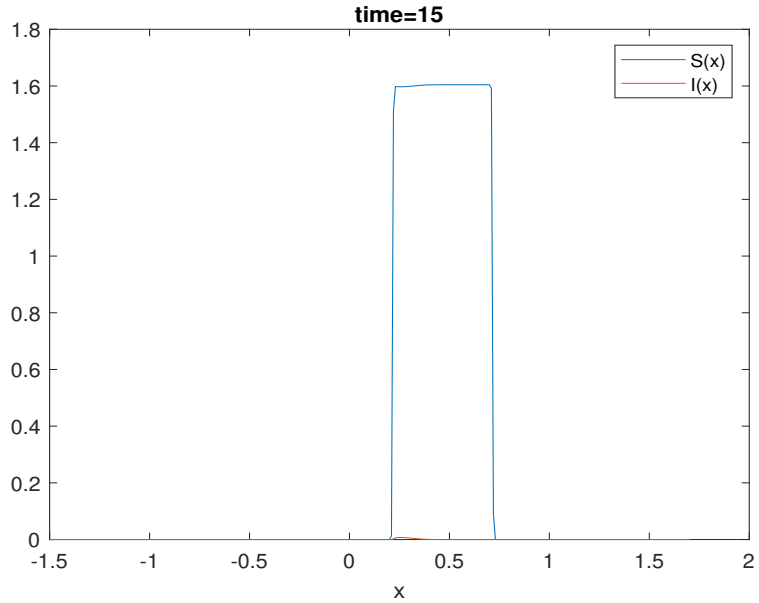
$$M \in \left(\frac{\alpha\gamma}{\beta}, +\infty\right)\,.$$

Numerical Simulations - Disease Free Equilibrium

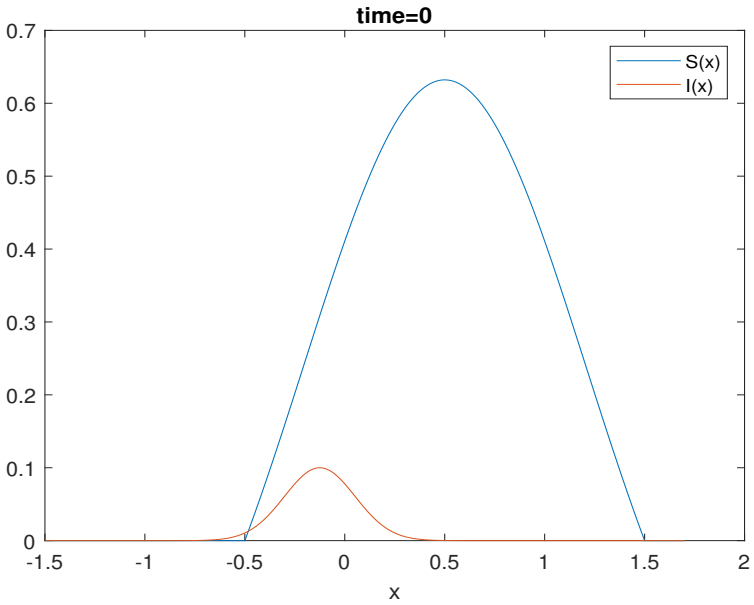


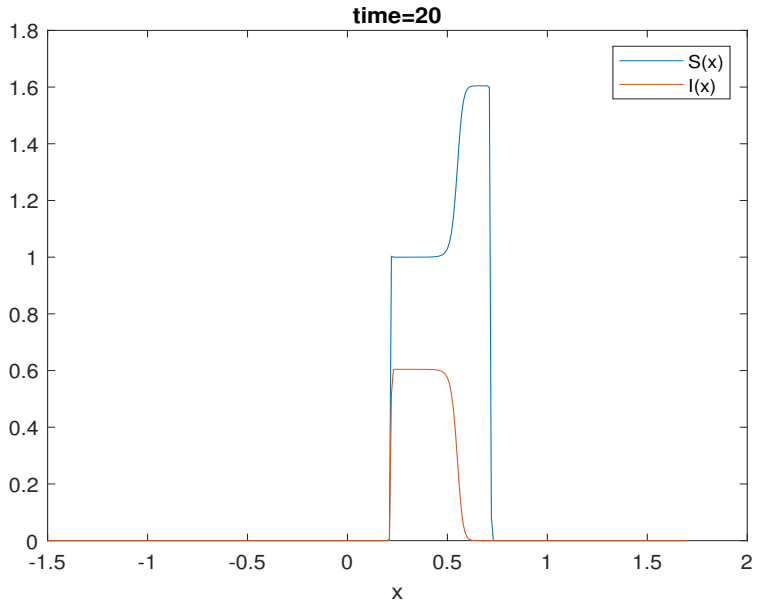


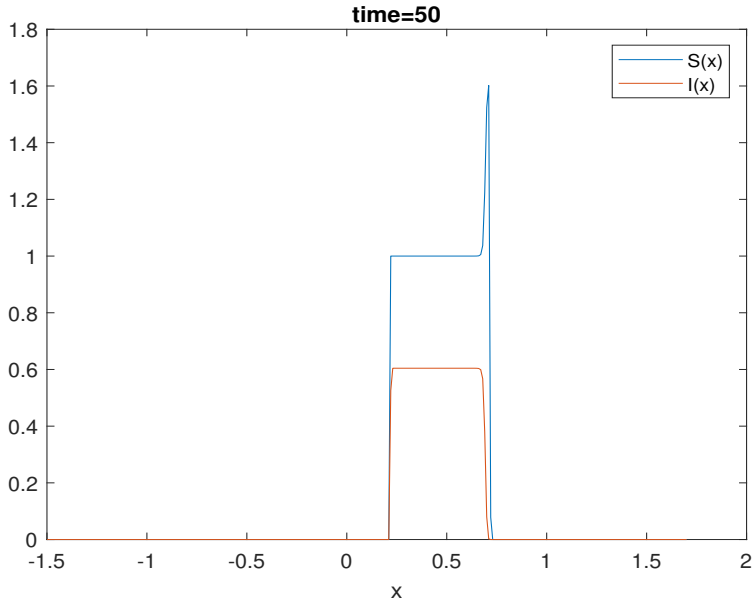


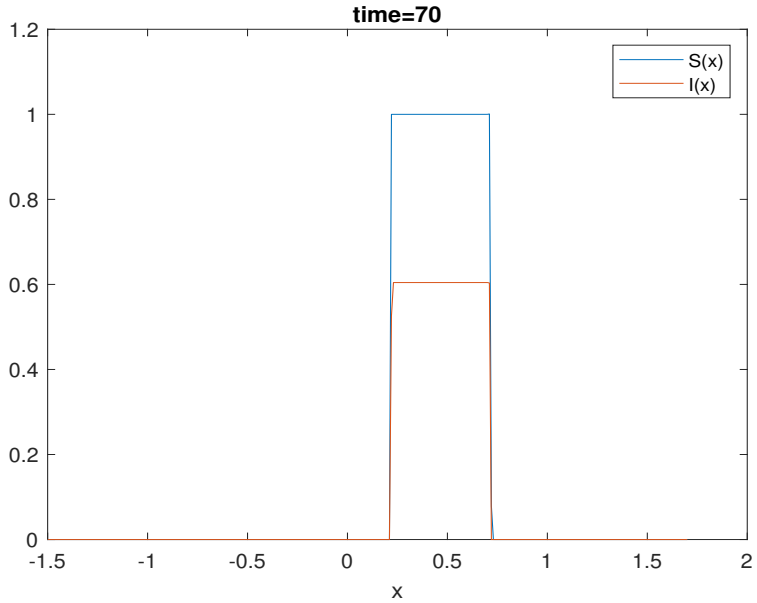


Numerical Simulations - Endemic Equilibrium









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THANK YOU FOR YOUR ATTENTION