Singularity formation in Mean Curvature Flow

A course in Geometric Flows,
Universidad de Granada, 2014

Version:
December 15, 2014

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1 Introduction to Mean Curvature Flow

Let M^n be closed, and $F : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a smooth family of immersions. Let $M_t := F(M,t)$. We call this family a Mean Curvature flow, MCF for short, starting at an initial immersion F_0, if

$$\begin{aligned} \frac{\partial F}{\partial t} &= -H \cdot \nu = \vec{H} \quad (= \Delta_{M_t} F) \\ F(\cdot, 0) &= F_0 \ , \end{aligned}$$

where $H = \lambda_1 + \cdots + \lambda_n$ and λ_i are the principal curvatures of M_t, i.e. the eigenvalues of the Weingarten map $D\nu : T_pM \to T_pM$, where ν is a choice unit normal vector field.

Remark 1.0.1: i) A sphere with initial radius R_0 evolves with radius $R(t) = \sqrt{R_0^2 - 2nt}$.

ii) The evolution equation for a surface, which is locally given as the graph of a function u, is

$$\dot{u} = \sqrt{1+|Du|^2}\ \mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \left(\delta^{ij} - \frac{D^i u\, D^j u}{1+|Du|^2}\right) D_i D_j u\,.$$

This is a quasilinear parabolic equation. The strong maximum principle for such equations implies:

- avoidance principle: If two mean curvature flows $(M_t^1)_{t\in[0,T)}$ and $(M_t^2)_{t\in[0,T)}$ are initially disjoint (and at least one of them is compact), i.e. $M_0^1 \cap M_0^2 = \emptyset$ then $M_0^1 \cap M_0^2 = \emptyset \ \ \forall\, t \in (0,T)$. Thus by i) the maximal existence time of any MCF starting with a closed initial M_0 is finite.

- If M_0 is embedded, then M_t is embedded for all t. In this case we will always choose ν to be the outward unit normal.

iii) Writing the evolving surfaces for t close to zero as normal graphs over the initial surface with height function v, one can show that v satisfies a similar quasilinear parabolic equation as above. Standard theory then guarantees short-time existence and uniqueness.

iv) Mean curvature flow is the L^2-gradient flow of the area functional, i.e. comparing with normal speeds with the same L^2-norm, the mean curvature is the speed which decreases the area of the surface the fastest.

In the following we denote by

$$g_{ij} = \left\langle \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j} \right\rangle$$

the induced metric on M_t, and its second fundamental form $A = \{h_{ij}\}$ by

$$h_{ij} = \left\langle \frac{\partial F}{\partial x_i}, \frac{\partial}{\partial x_j}\nu \right\rangle = -\left\langle \frac{\partial^2 F}{\partial x_j \partial x_i}, \nu \right\rangle.$$

The inverse of the metric g_{ij} will be denoted by g^{ij}. Then the matrix of the Weingarten map is given by $h^i_{\ j} = g^{il}h_{lj}$ and the mean curvature by $H = h^i_{\ i} = g^{ij}h_{ij} = \lambda_1 + \ldots + \lambda_n$. The square of the norm of the second fundamental form will be denoted by $|A|^2 = g^{mn}g^{st}h_{ms}h_{nt} = h^n_{\ s}h^s_{\ n} = \lambda_1^2 + \ldots + \lambda_n^2$. We denote the induced measure on M_t by $d\mu$.

Lemma 1.0.2. *The following evolution equations hold.*

$$\frac{\partial}{\partial t}\nu = \nabla H \qquad \frac{\partial}{\partial t}g_{ij} = -2Hh_{ij}$$

$$\frac{\partial}{\partial t}g^{ij} = 2Hh^{ij} \qquad \frac{\partial}{\partial t}d\mu = -H^2\, d\mu$$

$$\frac{\partial}{\partial t}h_{ij} = \Delta h_{ij} - 2Hh_{im}h^m_{\ j} + |A|^2 h_{ij} \qquad \frac{\partial}{\partial t}h^i_{\ j} = \Delta h^i_{\ j} + |A|^2 h^i_{\ j}$$

$$\frac{\partial}{\partial t}H = \Delta H + |A|^2 H \qquad \frac{\partial}{\partial t}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + 2|A|^4$$

For a proof of these evolution equations see [19].

In the following we will often make use of the maximum principle, which we state here for scalar functions and symmetric 2-tensors.

Theorem 1.0.3 (Strong maximum principle for parabolic equations)**.**
Let $f : M \times [0,T) \to \mathbb{R}$ satisfy

$$\frac{\partial f}{\partial t} \geq \Delta f + b^i \nabla_i f + c\, f$$

for some smooth funtions b^i, c, where $c \geq 0$. If $f(\cdot,0) \geq 0$ then

$$\min_M f(\cdot,t) \geq \min_M f(\cdot,0)\,.$$

Furthermore, if $f(p,t_0) = \min_M f(\cdot,0)$ for some $p \in M$, $t > 0$, then $f \equiv \min_M f(\cdot,0)$ for $0 \leq t \leq t_0$.

Theorem 1.0.4 (Strong maximum principle for symmetric 2-tensors (Hamilton))**.** *Let $m^i_{\ j}$ be a symmetric bilinear form, which solves*

$$\frac{\partial m^i_{\ j}}{\partial t} \geq \Delta m^i_{\ j} + \phi^i_{\ j}(m^i_{\ j})\,,$$

where $\phi^i_{\ j}$ is a symmetric bilinear form, depending on $m^i_{\ j}$, with the property $\phi^i_{\ j}(m^i_{\ j}) \geq 0$ if $m^i_{\ j} \geq 0$. If $m^i_{\ j} \geq 0$ for $t = 0$ then $m^i_{\ j} \geq 0$ for all $t \geq 0$. Furthermore, for $t > 0$, the rank of the null-space of $m^i_{\ j}$ is constant, and the null-space is invariant under parallel transport and invariant in time.

The evolution equations for H and $h^i_{\ j}$ thus imply

- If $H \geq 0$ at $t = 0$, then $H > 0$ for $t > 0$. Furthermore $\min_M H(\cdot,t)$ is strictly increasing in time.

- If M_0 is convex, then M_t is strictly convex for $t > 0$.

1.1 The maximal time of existence

We now want to show that the solution exists as long as $|A|$ stays bounded. To do this we first need the evolution equation of higher derivatives of A.

Lemma 1.1.1.

$$\frac{\partial}{\partial t}|\nabla^m A|^2 = \Delta|\nabla^m A|^2 - 2|\nabla^{m+1}A|^2 + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A * \nabla^m A$$

For a proof see [15].

With this we can show that all higher derivatives of A stay bounded if A is bounded.

Proposition 1.1.2. *If $|A|^2 \leq C_0$ on $M \times [0,T)$, then*

$$|\nabla^m A|^2 \leq C_m \qquad \text{on } M \times [0,T)\,,$$

where $C_m = C_m(n, M_0, C_0)$.

Proof. We have

$$\frac{\partial}{\partial t}|\nabla^m A|^2 \leq \Delta|\nabla^m A|^2 - 2|\nabla^{m+1}A|^2 + C(n,m) \sum_{i+j+k=m} |\nabla^i A|\cdot|\nabla^j A|\cdot|\nabla^k A|\cdot|\nabla^m A|\,.$$

We give a proof by induction. The case $m = 0$ is trivially true. So we assume that for $m > 0$ we have $|\nabla^l A|^2 \leq C_l$ for $0 \leq l \leq m-1$. Thus

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^{m-1} A|^2 &\leq \Delta|\nabla^{m-1} A|^2 - 2|\nabla^m A|^2 + B_{m-1} \\ \frac{\partial}{\partial t}|\nabla^m A|^2 &\leq \Delta|\nabla^m A|^2 + B_m(1 + |\nabla^m A|^2)\,.\end{aligned}$$

We consider the function $f := |\nabla^m A|^2 + B_m|\nabla^{m-1}A|^2$, which satisfies

$$\frac{\partial f}{\partial t} \leq \Delta f - B_m|\nabla^m A|^2 + B \leq \Delta f - B_m f + B'\,.$$

Thus $\tilde{f} := e^{B_m t} f - B' t$ satisfies

$$\frac{\partial \tilde{f}}{\partial t} \leq \Delta \tilde{f}$$

which implies $\tilde{f}(t) \leq \max_M \tilde{f}(0)$ and thus

$$f(t) \leq e^{-B_m t}(\max_M \tilde{f}(0) + tB') \leq C'$$

□

Let us assume from now on that $[0, T)$ is the maximal time of existence of the flow.

Corollary 1.1.3. *We have* $\limsup_{t \to T} \max_{M_t} |A|^2 = \infty$.

Proof. Let us assume to the contrary that $|A|^2 \leq C_0$ for $t \in [0, T)$. By theorem 1.1.2 all higher derivatives of A are bounded. This implies that $F(\cdot, t)$ converges smoothly to a limiting immersion $F(\cdot, T)$. But by short-time existence this implies that we can extend the solution further, which contradicts the assumption that T is maximal. □

One can even show that bounds on the second fundamental form imply local bounds for all higher derivatives.

Theorem 1.1.4 (Ecker/Huisken [9, 8])**.** *Let (M_t) be a smooth, properly embedded solution of MCF in $B_\rho(x_0) \times (t_0 - \rho^2, t_0)$ which satisfies the estimate*

$$|A(x)|^2 \leq \frac{C_0}{\rho^2}$$

for all $x \in M_t \cap B_\rho(x_0)$ and $t \in (t_0 - \rho^2, t_0)$. Then for every $m \geq 1$ there is a constant C_m, depending only on n, m and C_0 such that for all $x \in M_t \cap B_{\rho/2}(x_0)$ and $t \in (t_0 - \rho^2/4, t_0)$,

$$|\nabla^m A|^2 \leq \frac{C_m}{\rho^{2(m+1)}} \ .$$

1.2 The monotonicity formula

We consider for $t < t_0$

$$\rho_{x_0,t_0}(x,t) := \frac{1}{(4\pi(t_0-t))^{n/2}} e^{-\frac{|x-x_0|^2}{4(t_0-t)}} ,$$

which is the backward heat kernel in $\mathbb{R}^{n+1}$, based at (x_0,t_0) and scaled by a factor $(4\pi(t_0-t))^{1/2}$. G. Huisken discovered the following monotonicity property under MCF:

$$\frac{d}{dt}\int\limits_{M_t} \rho_{x_0,t_0}\, d\mu = -\int\limits_{M_t} \Big|\vec{H} + \frac{(x-x_0)^\perp}{2(t_0-t)}\Big|^2 \rho_{x_0,t_0}\, d\mu \qquad \text{for } t < t_0\,.$$

We will see that this implies that close to a singularity the flow is nearly self-similar - that is it is nearly moving only by homotheties. This can be seen as follows. For $\lambda > 0$ consider the following parabolic rescaling of the flow

$$F_\lambda(\cdot,t) := \lambda\big(F(\cdot,\lambda^{-2}t + t_0) - x_0\big) \tag{1.1}$$

where $t \in [-\lambda^2 t_0, \lambda^2(T-t_0))$. This corresponds to translating the point (x_0,t_0) to $(0,0)$ in space-time, and rescaling space by a factor λ and time by a factor λ^2. This again constitutes a mean curvature flow. Let

$$\Theta_t(x_0,t_0) := \int\limits_{M_t} \rho_{x_0,t_0}\, d\mu\,,$$

which is decreasing in time and thus the limit

$$\lim_{t\nearrow t_0} \Theta_t(x_0,t_0) =: \Theta(x_0,t_0)$$

exists and is called the Gaussian density at (x_0,t_0). The monotonicity formula implies

$$\Theta_t(x_0,t_0) - \Theta(x_0,t_0) = \int\limits_t^{t_0}\int\limits_{M_{t'}} \Big|\vec{H} + \frac{(x-x_0)^\perp}{2(t_0-t')}\Big|^2 \rho_{x_0,t_0}\, d\mu\, dt'\,.$$

We now consider a sequence $\lambda_i \to \infty$ of rescalings, and take a fixed time $t_1 \in [-\lambda_i^2 t_0, 0)$. One can check then that the last formula implies

$$\Theta_{t_0+\lambda_i^{-2}t_1}(x_0,t_0) - \Theta(x_0,t_0) = \int_{t_1}^{0} \int_{M_{t'}^{\lambda_i}} \left|\vec{H} - \frac{(x)^\perp}{2t'}\right|^2 \rho_{0,0}\, d\mu\, dt' \,. \tag{1.2}$$

If one assumes that $(M_t^{\lambda_i})$ converges smoothly to a limiting mean curvature flow (M_t^∞), defined on $(-\infty, 0)$, then the above formula implies that M_t^∞ satisfies

$$\vec{H} = \frac{x^\perp}{2t} \quad \text{i.e.} \quad H = \frac{\langle x, \nu\rangle}{-2t}$$

for $t < 0$. This implies that the solution is self-similar, i.e. it only moves by scaling

$$M_t = \sqrt{-t} \cdot M_{-1} \,.$$

Remark 1.2.1: i) One calls a singularity at time T of type I, if we one has the bound

$$\sup_{M_t} |A|^2 \leq \frac{C}{T-t}$$

for some C. Doing a parabolic rescaling around a point (x_0, T) as above one sees that this bound is scaling invariant, and one can show that the flows converge subsequentially (at least locally) to a smooth limiting flow, which is self-similarly shrinking.

ii) Singularities which do not satisfy this bound are called type II singularities. Even in this case, one can still show that one can extract a weak limit, where the limiting object is not a smooth MCF anymore, but a so-called Brakke-flow. A Brakke flow is a family of moving varifolds, which satisfies MCF in an integrated sense.

iii) The above monotonicity formula implies that the Gaussian density is upper semicontinous in (x_0, t_0) and also if one considers converging sequences of flows. In the case that we have a sequence of rescalings as above, the limit is self-similar with $\Theta_t(0,0)$ constant in time and equal to the initial Gaussian density $\Theta(x_0, t_0)$. Since at every smooth point of the flow $\Theta = 1$, we have that at each singular point $\Theta \geq 1$. Similarly, any point reached by the flow has $\Theta \geq 1$.

For $n \geq 2$ one can classify all self-similar solutions, which habe non-negative mean curvature.

Theorem 1.2.2 (Huisken [16, 17], Colding/Minicozzi [7]). *If M^n, for $n \geq 2$, is an embedded hypersurface in $\mathbb{R}^{n+1}$, with non-negative mean curvature, satisfying $H = \langle x, \nu \rangle / 2$, then M^n is of the form*

$$\mathbb{S}^{n-m}_{(2(n-m))^{1/2}} \times \mathbb{R}^m$$

for $m = 0, \ldots, n$.

This has deep implications for the structure of singularities of mean curvature flow of mean convex surfaces. For curves no condition is needed:

Theorem 1.2.3 (Abresch/Langer [1]). *The only embedded, closed curves in $\mathbb{R}^2$ satisfying $k = \langle x, \nu \rangle / 2$ are either a straight line through the origin or the circle of radius $\sqrt{2}$.*

2 Evolution of closed curves in the plane

In this section we consider the evolution of closed curves in the plane under curve shortening flow, that is mean curvature flow in $\mathbb{R}^2$. The evolution equation for a smooth family of curve (γ_t) is then given by

$$\frac{d\gamma_t}{dt} = \vec{k}\,,$$

where $\vec{k}$ is the curvature vector of the curve. In the following we want to present a proof of the following two theorems.

Theorem 2.0.4 (Gage/Hamilton [11])**.** *Simple, closed, convex curves remain convex under curve shortening flow and shrink in finite time to a 'round' point.*

Theorem 2.0.5 (Grayson [12])**.** *Under curve shortening flow, simple, closed curves become convex in finite time.*

Grayson's argument, following the work of Gage and Hamilton for convex curves, is rather delicate. More recently the proof has been simplified by using isoperimetric estimates to rule out certain types of singularities: Huisken [18] proved an estimate, controlling the ratio between the intrinsic and extrinsic distance between two points on the evolving curve, and Hamilton [14] gave an estimate controlling the ratio of the isoperimetric profile to that of a circle of the same area. The proof then follows in both cases by destinguishing type I and type II singularities. In the first case one can use Huisken's monotonicity formula and the classification of self-similarly shrinking solutions to show that the asymptotic shape of the solution is the shrinking circle. In the case of a type II singularity one can do a rescaling

to produce a convex limiting curve, which by Hamilton's Harnack estimate [13] has to be the 'grim reaper' curve. But this violates the isoperimetric bound ruling out singularities of type II. Very recently, using a refined isoperimetric estimate, a very elegant and direct proof of Grayson's result has been given by Andrews and Bryan [3] which does not use Huisken's monotonicity formula or the classification of singularities.

In the following we will present Huisken's estimate on the ratio between the extrinsic and intrinsic distance. Using the monotonicity formula we then show that one can give a proof of Grayson's result by using only Huisken's monotonicity formula and the classification of self-similarly shrinking curves, thus avoiding the analysis of type II singularities.

2.1 Intrinsic versus extrinsic distance

In this section we will present Huisken's proof that given two points p, q on γ_t, the ratio between the intrinsic distance along the curve and the extrinsic distance stays controlled under curve shortening flow. We follow the original article [18].

Let $F : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ be a closed, embedded curve moving by curve shortening flow. Let $L(t)$ be the total length of the curve, and l be the intrinsic distance between two points, which is defined for $0 \leq l \leq L/2$. Let the smooth function $\psi : \mathbb{S}^1 \times \mathbb{S}^1 \times [0, T) \to \mathbb{R}$ be given by

$$\psi := \frac{L}{\pi} \sin\left(\frac{l\pi}{L}\right).$$

Note that since $\sin\left(\frac{l\pi}{L}\right) = \sin\left(\frac{(L-l)\pi}{L}\right)$ the function is smooth at points (p, q) with $l(p, q) = L/2$. Then the ratio d/ψ, where $d(p, q) = |F(p) - F(q)|$ is the extrinsic distance between two points on the curve, is equal to 1 on the diagonal of $\mathbb{S}^1 \times \mathbb{S}^1$ for any smooth embedding of $\mathbb{S}^1 \to \mathbb{R}^2$ and the ratio d/ψ is identically 1 on any round circle.

Theorem 2.1.1 (Huisken)**.** *The minimum of d/ψ on $\mathbb{S}^1 \times \mathbb{S}^1$ is non-decreasing under curve shortening flow.*

Proof. Since d/ψ is equal to one on the diagonal, it is sufficient to show that whenever d/ψ has a spatial minimum $(d/\psi)(p,q,t) < 1$, at some pair of points $(p,q) \in \mathbb{S}^1 \times \mathbb{S}^1$, and some time $t_0 \in [0,T)$, then

$$\frac{d}{dt}(\psi/d)(p,q,t_0) > 0\,.$$

We take s to be the arclenght parameter at time t_0, and without loss of generality $0 \leq s(p) < s(q) \leq L(t_0)/2$, such that $l(p,q) = s(q) - s(p)$. For any variation $\xi \in T_p\mathbb{S}^1_{t_0} \oplus T_q\mathbb{S}^1_{t_0}$ we have that the for the first and second variation

$$\delta(\xi)(d/\psi)(p,q,t_0) = 0 \qquad \delta^2(\xi)(d/\psi)(p,q,t_0) \geq 0$$

From the vanishing of the first variation for $\xi = e_1 \oplus 0$ and $\xi = 0 \oplus e_2$ one can easily compute that

$$\langle \omega, e_1 \rangle = \langle \omega, e_2 \rangle = \frac{d}{\psi} \cos\left(\frac{l\pi}{L}\right), \tag{2.1}$$

where $e_1 = \frac{\partial}{\partial s}F(p,t_0)$, $e_2 = \frac{\partial}{\partial s}F(q,t_0)$ and $\omega = -d^{-1}(p,q,t_0)\big(F(p,t_0) - F(q,t_0)\big)$. This implies two possibilities.

Case 1: $e_1 = e_2$. Choosing $\xi = e_1 \oplus e_2$ in the second variation inequality we can compute that

$$0 \leq \delta^2(e_1 \oplus e_2)(d/\psi) = \frac{1}{\psi}\langle \omega, \vec{k}(q,t_0) - \vec{k}(p,t_0)\rangle \tag{2.2}$$

Case 2: $e_1 \neq e_2$. Using that in this case $e_1 + e_2$ is parallel to ω and using the second variation inequality with $\xi = e_1 \ominus e_2$ one can deduce that

$$0 \leq \delta^2(e_1 \ominus e_2)(d/\psi) = \frac{1}{\psi}\langle \omega, \vec{k}(q,t_0) - \vec{k}(p,t_0)\rangle + \frac{4\pi^2}{L^2}\frac{d}{\psi}\,. \tag{2.3}$$

We can now estimate $(d/dt)(d/\psi)$. Using the original evolution equation and that

$(d/dt)(ds) = -k^2(ds)$ we see

$$\begin{aligned}\frac{d}{dt}\Big(\frac{d}{\psi}\Big) &= \frac{1}{d\psi}\big\langle F(q,t_0) - F(p,t_0), \vec{k}(q,t_0) - \vec{k}(p,t_0)\big\rangle - \frac{d}{\psi^2}\frac{d}{dt}\Big(\frac{L}{\pi}\sin\Big(\frac{l\pi}{L}\Big)\Big) \\ &= \frac{1}{\psi}\big\langle \omega, \vec{k}(q,t_0) - \vec{k}(p,t_0)\big\rangle + \frac{d}{\psi^2\pi}\sin\alpha \int\limits_{\mathbb{S}^1} k^2\,ds \\ &\quad + \frac{d}{\psi^2}\cos\alpha \int\limits_p^q k^2\,ds - \frac{dl}{\psi^2 L}\cos\alpha \int\limits_{\mathbb{S}^1} k^2\,ds\,,\end{aligned}$$

where we introduced $\alpha = l\pi/L,\ 0 < \alpha \le L/2$. Again we distinguish the two cases from above:

Case 1: $e_1 = e_2$. Using (2.2) we see

$$\begin{aligned}\frac{d}{dt}\Big(\frac{d}{\psi}\Big) &\ge \frac{d}{\psi L}\int\limits_{\mathbb{S}^1} k^2\,ds + \frac{d}{\psi^2}\cos\alpha \int\limits_p^q k^2\,ds - \frac{dl}{\psi^2 L}\cos\alpha \int\limits_{\mathbb{S}^1} k^2\,ds \\ &= \frac{d}{\psi L}\big(1 - \frac{l}{\psi}\cos\alpha\big)\int\limits_{\mathbb{S}^1} k^2\,ds + \frac{d}{\psi^2}\cos\alpha \int\limits_p^q k^2\,ds\ .\end{aligned}$$

Now note that $\frac{l}{\psi}\cos\alpha = \alpha(\tan\alpha)^{-1} < 1$, since $0 < \alpha \le \pi/2$ by assumption. This shows the desired sign on the time derivative.

Case 2: $e_1 \ne e_2$. By (2.3) we have

$$\frac{d}{dt}\Big(\frac{d}{\psi}\Big) \ge -\frac{4\pi^2}{L^2}\frac{d}{\psi} + \frac{d}{\psi L}\big(1 - \frac{l}{\psi}\cos\alpha\big)\int\limits_{\mathbb{S}^1} k^2\,ds + \frac{d}{\psi^2}\cos\alpha \int\limits_p^q k^2\,ds\ .$$

Since $\int_{\mathbb{S}^1} k\,ds = 2\pi$ the Hölder inequality gives

$$\int\limits_{\mathbb{S}^1} k^2\,ds \ge \frac{1}{L}\Big(\int\limits_{\mathbb{S}^1} k\,ds\Big)^2 = \frac{4\pi^2}{L},$$

and since as before $\left(1 - \frac{l}{\psi}\cos\alpha\right) > 0$ we obtain

$$\frac{d}{dt}\left(\frac{d}{\psi}\right) \geq \frac{d}{\psi^2 l}\cos\alpha\left(- \frac{4\pi^2 l^2}{L^2} + l\int_p^q k^2\, ds\right) . \tag{2.4}$$

But now notice that

$$l\int_p^q k^2\, ds \geq \left(\int_p^q k\, ds\right)^2 = \beta^2$$

where $0 < \beta \leq \pi$ is the angle between e_1 and e_2. Since $e_1 + e_2$ is parallel to ω we have by equation (2.1)

$$\cos\left(\frac{\beta}{2}\right) = \langle e_1, \omega\rangle = \langle e_2, \omega\rangle = \frac{d}{\psi}\cos\alpha$$

and since by assumption $(d/\psi)(p, q, t_0) < 1$ we have $\cos(\beta/2) < \cos\alpha$ and thus $\alpha < \beta/2$. Thus

$$l\int_p^q k^2\, ds \geq \beta^2 > 4\alpha^2 = \frac{4\pi^2 l^2}{L^2} ,$$

which implies the desired inequality. □

Remark 2.1.2: Note that this implies that $\min_{\gamma_t} \frac{d}{\psi} \geq \min_{\gamma_0} \frac{d}{\psi} > 0$ for all $t \in [0, T)$ since we assume that the initial curve is embedded. It is also important to note that this quantity is invariant under scaling.

2.2 Convergence to a 'round' point

The distance comparison principle from the last section will enable us to show that if (x_0, T) is a singular point of the flow, then any sequence of rescaling as in (1.1) converges to the homothetically shrinking circle.

We will first show the following weaker convergence result.

Lemma 2.2.1. *Let (x_0, T) be a point reached by the flow. Then for any sequence of rescalings as in* (1.1) *with $\lambda_i \to \infty$ there exists a subsequence, labeled again the*

same, such that for almost all $t \in (\infty, 0)$ *and for any* $\alpha \in (0, 1/2)$

$$\gamma_t^{\lambda_i} \to \bar{\gamma}_t$$

in $C^{1,\alpha}_{loc}$*, where* $\bar{\gamma}_t$ *is either a constant line through the origin or the self-similarly shrinking circle.*

Proof. Let

$$f_i(t) := \int\limits_{\gamma_t^{\lambda_i}} \left| \vec{k} - \frac{x^\perp}{2t} \right|^2 \rho_{0,0}(\cdot, t)\, ds.$$

Note that (1.2) implies that $f_i \to 0$ in $L^1_{\text{loc}}((-\infty, 0])$. Thus there exists a subsequence such that f_i converges point-wise a.e. to zero. This implies that for any such t' and $R > 0$

$$\int\limits_{\gamma_t^{\lambda_i} \cap B_R(0)} |k|^2 ds \leq C\,,$$

independent of i. By choosing a further subsequence we can assume that γ_t^i converges in $C^{1,\alpha}_{\text{loc}}$ to a limiting curve. Note first that the distance comparison principle from the last section implies that no higher multiplicities can develop, and the limiting curve is embedded. Note further that each limiting curve is in $W^{2,2}_{\text{loc}}$ and is a weak solution of

$$\vec{\kappa} = \frac{x^\perp}{2t}\,.$$

By elliptic regularity, each such curve is actually smooth, and thus by theorem 1.2.3 the limiting curve is either a straight line through the origin or the centered circle of radius $\sqrt{-2t}$. □

Note that the Gaussian density of a line through the origin is one, and the Gaussian density of the shrinking circle can be computed to be $\sqrt{2/e} \approx 1.520$. Since the Gaussian density of the limiting flow coincides with the Gaussian density of the initial flow at the point (x_0, T) we only have the following two cases. Either any rescaling subconverges to a line through the origin or all rescalings converge to the shrinking circle - independently of the sequence of rescalings chosen.

Let us first consider the case that $\Theta(x_0, T) = 1$, so any sequence of rescalings has

a subsequence which converges for a.e. t to a line through the origin. Note that by using big spheres as barriers we see that the orientation of the limiting line does not depend on t. We can assume w.l.o.g. that the limiting line is the axis $\{x_2 = 0\}$. Again by using spheres as barriers we can actually see that for $\varepsilon > 0$ there exists i_0 such that for $i > i_0$ we have

$$\gamma_t^{\lambda_i} \cap B_{100}(0) \subset \{|x_2| \leq \varepsilon\} \cap B_{100}(0) \qquad \text{for all } t \in [-2, 0).$$

We now fix such an $i > i_0$. We want to show that (x_0, t_0) is a smooth point of the flow, that is, there is a $C > 0$ such that

$$|k|_{\gamma_t^{\lambda_{i_0}}}\Big|_{B_1(0)} \leq C \qquad \text{for all } t \in [-1, 0).$$

(Note that this implies by theorem 1.1.4 that γ_t is smooth in a neighbourhood of (x_0, T) and thus any sequence converges to the same line through the origin).

By the previous lemma, we can assume that there is a $t_0 \in (-3, -2)$ such that $\gamma_{t_0}^{\lambda_i}$ is $C^{1,\alpha}$-close to $\{x_2 = 0\}$ on $B_{100}(0)$. This implies that $\gamma_{t_0}^{\lambda_i}$ can be written as a graph of a function with bounded gradient over $\{x_2 = 0\}$ on $B_{50}(0)$. A deep result by S. Angenent [4, 5, 6] states that the number of intersections between two graphical solutions of curve shortening flow is decreasing in time. This implies, by comparing with tilted static lines, that $\gamma_{t_0}^{\lambda_i}$ remains a graph with bounded gradient in $B_{10}(0)$ over $\{x_2 = 0\}$ for all $t \in (t_0, 0)$. The interior estimates for graphical mean curvature flow of Ecker/Huisken [10] then imply the claimed bound on the curvature.

Alternatively one can use that due to the $C^{1,\alpha}$-convergence, the Gaussian densities at t_0

$$\Theta_{t_0}(x, t_0 + r^2) \leq 1 + \varepsilon$$

for all $0 < r < r_0$ and $x \in B_{50}(0)$. A regularity theorem for mean curvature Flow due to B. White [20] then states that for $\varepsilon \leq \varepsilon_0 = \varepsilon_0(n)$ the second fundamental form (and all its derivatives) are bounded on $B_{10}(0) \times [t_0 + 1, t_0 + r_0^2/2]$. In our case we can assume that $r_0 \geq 3$, which yields the desired bound.

Note that by the previous reasoning there has to exist a point (x_0, T) such that

every rescaling sequence has a subsequence which converges point-wise a.e. to the shrinking circle. Note that we thus can assume that for every $\varepsilon > 0$ there is a $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$ there exits a $t_\lambda \in (-3, -2)$ such that

$$\gamma^\lambda_{t_\lambda} \text{ is } \varepsilon\text{-close to } \sqrt{-2t_\lambda} \cdot \mathbb{S}^1 \text{ in } C^{1,\alpha} . \tag{2.5}$$

By using shrinking spheres as barriers, this implies that for ε small enough

$$\gamma^\lambda_t \subset B_{(1+2\varepsilon)\sqrt{-2t}}(0) \setminus B_{(1-2\varepsilon)\sqrt{-2t}}(0) \qquad \text{for all } t \in (-2, -1) .$$

This implies that on $(-2, -1)$ the subsequence converges in Hausdorff distance to the shrinking circle. But since every rescaling sequence has such a subsequence, this implies that for every sequence the flow converges on $(-2, -1)$ in Hausdorff distance to the shrinking circle. This already implies that the rescaled curves

$$(T - t)^{-1/2}\big(\gamma_t - x_0\big) \to \mathbb{S}^1_{\sqrt{2}} \tag{2.6}$$

in Hausdorff distance.

To prove higher order convergence one can use (2.5) together with the regularity result of B. White as described above. We want to present an alternative way here. To do this note that (2.5) implies that for ε small enough, $\gamma^\lambda_{t_\lambda}$ is strictly star-shaped with respect to the origin, i.e. $\langle x, \nu\rangle > 0$. We fix such a $\lambda > \lambda_0$. The quantity $\langle x, \nu\rangle$ satisfies the following evolution equation

$$\frac{d}{dt}\langle x, \nu\rangle = \Delta\langle x, \nu\rangle + \langle x, \nu\rangle k^2 - 2k \tag{2.7}$$

and we also recall the evolution equation of the curvature

$$\frac{d}{dt}k = \Delta k + k^3 .$$

Thus the function $f := \langle x, \nu\rangle + 2(t - t_\lambda)k$ satisfies the evolution equation

$$\frac{d}{dt}f = \Delta f + fk^2$$

and by the maximum principle, since $f(t_\lambda) \geq 0$, we have $f \geq 0$ for $t \geq t_\lambda$ on (γ^λ_t).

This implies

$$k \geq -\frac{\langle x, \nu \rangle}{2(t - t_\lambda)} \geq -C_0 \qquad \text{on} (-2, 0)$$

Note that this one-sided curvature bound, together with (2.6), implies already C^1-convergence of the rescaled curves. Thus we can assume that for $t > -2$ the curves γ_t^λ are strictly star-shaped with respect to the origin, i.e.

$$\left\langle \frac{x}{|x|}, \nu \right\rangle \geq \delta > 0 \qquad \text{for all } -2 < t < 0$$

Together with (2.6) this implies

$$\langle x, \nu \rangle \geq 2\beta > 0 \,,$$

where we can choose $\beta = \delta\sqrt{-t'}/2$ on $(-2, t')$. Now note that

$$\begin{aligned} \frac{d}{dt}(k + 2C_0) = \Delta k + k^3 &= \Delta(k + 2C_0) + (k + 2C_0)k^2 - 2C_0 k^2 \\ &\leq \Delta(k + 2C_0) + (k + 2C_0)k^2 \,. \end{aligned}$$

This implies, together with (2.7), that the function

$$v := \frac{k + 2C_0}{\langle x, \nu \rangle - \beta}$$

satisfies

$$\frac{dv}{dt} \leq \Delta v + \frac{2}{\langle x, \nu \rangle - \beta} \big\langle \nabla \langle x, \nu \rangle, \nabla v \big\rangle + \Big(2 - \beta \frac{k^2}{k + 2C_0}\Big) v^2 \,.$$

on $(-2, t')$. Thus by the maximum principle we get a contradiction if v reaches a new maximum C_1 at $t \in (-2, t')$ which is bigger than $\max_{t=-2} v$ and

$$2 - \beta \frac{k^2}{k + 2C_0} < 0 \tag{2.8}$$

Note that $v \geq C_1$ implies that at that point

$$k \geq \beta C_1 - 2C_0 \,.$$

One can show that this implies (2.8) if

$$C_1 \geq \frac{4}{\beta^2} + \frac{2C_0}{\beta} \ .$$

This contradiction in turn implies

$$k + 2C_0 \leq \Big(\frac{4}{\beta^2} + \frac{2C_0}{\beta}\Big)\big(\langle x, \nu\rangle - \beta\big) \leq \Big(\frac{4}{\beta^2} + \frac{2C_0}{\beta}\Big)2\sqrt{-t} \ .$$

This yields with $\beta = \delta\sqrt{-t'}/2$ that

$$\sup_{t'} k \leq \frac{64}{\delta^2\sqrt{-t'}} + C_2$$

With the bound on k from below this implies that

$$|k| \leq \frac{C}{\sqrt{-t}} \ ,$$

which is a scaling invariant estimate. Thus by theorem 1.1.4 we have that the rescaled flows γ_t^λ converge in C^∞ to the shrinking circle on $(-2, -1)$. This again implies that the convergence in (2.6) is in C^∞.

Remark 2.2.2: One can show that the convergence in (2.6) is exponential. That is if one considers a new time variable $\tau = -\log(-t)$ then the convergence in (2.6) is actually exponential in every C^k-norm.

3 Evolution of closed, convex hypersurfaces

In this section we will study the evolution of closed, convex hypersurfaces in Euclidean space. We will present a proof Huisken's result below, where we do not follow the original proof, but again make use of the monotonicity formula and and estimate on the inner and outer radius of pinched, convex hypersurfaces by B. Andrews [2].

Theorem 3.0.3 (Huisken [15])**.** *Any closed, convex hypersurface becomes immediately strictly convex under MCF and converges in finite time to a 'round' point.*

The proof we present here is considerably shorter than Huisken's original proof. The idea of the proof is similar to the work of B. Andrews [2] but we shorten the proof further by using Huisken's classification of mean convex self-similar solutions.

We have seen in the introduction that Hamilton's maximum principle implies that closed, convex hypersurfaces stay convex and become immediately strictly convex. Enclosing the initial hypersurface by a big sphere and applying the avoidance principle we see that any such solution can only exist on a finite time interval. Let $F : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be such a maximal solution for $n \geq 2$.

3.1 Pinching of the principal curvatures

We can assume w.l.o.g. that M_0 is strictly convex. Since M_0 is compact there exists an $\varepsilon > 0$ such that

$$m^i_j := h^i_j - \varepsilon H \delta^i_j \geq 0 \ .$$

Lemma 3.1.1. *If initially $h^i_j - \varepsilon H \delta^i_j \geq 0$, then this is preserved under the flow.*

Proof. Using the evolution equation for h^i_j and H we see that

$$\frac{d}{dt} m^i_j = \Delta m^i_j + |A|^2 m^i_j \ .$$

Thus by the maximum principle $m^i_j \geq 0$ is preserved. □

This implies that at every point $p \in M$ it holds

$$\lambda_{\min}(p,t) \geq \varepsilon H(p,t) \geq \varepsilon \lambda_{\max}(p,t) \ ,$$

i.e. the principal curvatures are pinched. We define the inner radius ρ_- and the outer radius ρ_+ by

$$\begin{aligned} \rho_+(t) &= \inf\{r : B_r(y) \text{ encloses } M_t \text{ for some } y \in \mathbb{R}^{n+1}\} \\ \rho_-(t) &= \sup\{r : B_r(y) \text{ is enclosed by } M_t \text{ for some } y \in \mathbb{R}^{n+1}\} \, . \end{aligned}$$

A Lemma by B. Andrews relates the pinching of the principal curvatures to a pinching of the inner and outer radius.

Lemma 3.1.2 (Andrews, [2], lemma 5.1)**.** *Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be a closed, convex hypersurface such that there exists $0 < \varepsilon \leq 1$ such that at every point $p \in \Sigma$*

$$\lambda_{min}(p) \geq \varepsilon \lambda_{max}(p) \ .$$

Then there exits a $C_1 = C_1(\varepsilon, n) > 0$ *such that*

$$\rho_+ \leq C_1 \rho_- \ .$$

3.2 Convergence to a 'round' point

We will first show that the solution exists as long as it bounds a ball of radius $\delta > 0$. Note that since all the surfaces are convex we have $|A|^2 \leq H^2$.

Proposition 3.2.1. *Assume that* M_t *encloses* $B_\delta(0)$ *for* $t \in [0, t']$. *Then*

$$H(t) \leq 2\rho_+(t) \max\Big\{\frac{8n}{\delta^2}, \frac{2\sup_{M_0} H}{\delta}\Big\}, .$$

Proof. Since all M_t enclose $B_\delta(0)$ for $t \in [0, t']$ and are convex, we have

$$\langle x, \nu\rangle \geq \delta \ .$$

The evolution equation of $\langle x, \nu\rangle$ is given by

$$\frac{d}{dt}\langle x, \nu\rangle = \Delta\langle x, \nu\rangle + |A|^2\langle x, \nu\rangle - 2H \ .$$

Let $\beta = \delta/2$, then we have $\langle x, \nu\rangle - \beta \geq \beta$. As before we define the function

$$v = \frac{H}{\langle x, \nu\rangle - \beta}$$

which satisfies, using that $|A|^2 \geq \frac{1}{n}H^2$

$$\begin{aligned}\frac{dv}{dt} &= \Delta v + \frac{2}{\langle x, \nu\rangle - \beta}\big\langle \nabla\langle x, \nu\rangle, \nabla v\big\rangle + 2v^2 - \beta\frac{|A|^2}{H}v^2 \\ &\leq \Delta v + \frac{2}{\langle x, \nu\rangle - \beta}\big\langle \nabla\langle x, \nu\rangle, \nabla v\big\rangle + \big(2 - \frac{\beta}{n}H\big)v^2\end{aligned}$$

Let us assume that v attains a new maximum which is greater than C at a point

(p,t). Then we have at this point $H > \beta C$ and we get a contradiction if

$$C \geq \frac{2n}{\beta^2} \, .$$

Thus we obtain

$$H \leq \max\left\{\frac{2n}{\beta^2}, \frac{\sup_{M_0} H}{\beta}\right\}\big(\langle x, \nu\rangle - \beta\big) \leq 2\rho_+(t)\max\left\{\frac{2n}{\beta^2}, \frac{\sup_{M_0} H}{\beta}\right\}$$

□

By the previous proposition, together with lemma 3.1.2, we see that the solution exists until $\rho_- \to 0$. Furthermore the solution contracts for $t \to T$ to a point x_0.

Lemma 3.2.2 (Andrews, [2])**.** *We have with C_1 as in lemma 3.1.2:*

$$C_1^{-1}\sqrt{2n(T-t)} \leq \rho_-(t) \, .$$

Proof. Let y be such that $\mathbb{S}_{\rho_+(t)}(y)$ encloses M_t. By the avoidance principle $M_{t'}$ remains enclosed by $\mathbb{S}_{\rho(t)}(y)$ for all t in the range (t', T), where $\rho(t') = \sqrt{\rho_+^2(t) - 2n(t'-t)}$. Thus

$$\rho_+^2(t') \leq \rho_+^2(t) - 2n(t'-t) \, .$$

Since the solution exists until $t' = T$ we have

$$\rho_+^2(t) \geq 2n(T-t) \;\Rightarrow\; \rho_-^2(t) \geq C_2^{-2}\, 2n(T-t) \, .$$

□

Applying this to proposition before, with $\beta = \rho_-(t')/2$, we see that we have

$$|A|(t') \leq H(t') \leq \frac{C}{\sqrt{T-t'}} \, .$$

Again, this is a scaling invariant estimate. Thus the rescalings as in (1.1) converge smoothly on any compact sub-interval of $(-\infty, 0)$ to a convex, selfsimilar solution.

Note that the limiting solution still satisfies $\rho_+ \leq C_1 \rho_-$ and thus it can only be the shrinking sphere by Huisken's classification. Since this is true for any sequence of rescalings, we obtain that

$$\frac{1}{\sqrt{T-t}}\big(F(\cdot,t) - x_0\big) \to \mathbb{S}^n_{\sqrt{2n}}$$

in C^∞.

Remark 3.2.3: As before for curves one can also show that the convergence is exponential.

Bibliography

[1] Uwe Abresch and Joel Langer, *The normalized curve shortening flow and homothetic solutions*, J. Differential Geom. **23** (1986), no. 2, 175–196.

[2] Ben Andrews, *Contraction of convex hypersurfaces in Euclidian space*, Calc. Var. **2** (1994), 151–171.

[3] Ben Andrews and Paul Bryan, *Curvature bound for curve shortening flow via distance comparison and a direct proof of Grayson's theorem*, J. Reine Angew. Math. **653** (2011), 179–187.

[4] Sigurd Angenent, *The zero set of a solution of a parabolic equation*, J. Reine Angew. Math. **390** (1988), 79–96.

[5] ———, *On the formation of singularities in the curve shortening flow*, J. Differential Geom. **33** (1991), no. 3, 601–633.

[6] ———, *Parabolic equations for curves on surfaces. II. Intersections, blow-up and generalized solutions*, Ann. of Math. (2) **133** (1991), no. 1, 171–215.

[7] Tobias H. Colding and William P. Minicozzi, II, *Generic mean curvature flow I: generic singularities*, Ann. of Math. (2) **175** (2012), no. 2, 755–833.

[8] Klaus Ecker, *Regularity theory for mean curvature flow*, Progress in Nonlinear Differential Equations and their Applications, 57, Birkhäuser Boston Inc., Boston, MA, 2004.

[9] Klaus Ecker and Gerhard Huisken, *Interior estimates for hypersurfaces moving by mean curvature*, Invent. Math. **105** (1991), no. 3, 547–569.

[10] ———, *Interior estimates for hypersurfaces moving by mean curvature*, Invent. Math. **105** (1991), no. 3, 547–569.

[11] M.E. Gage and R.S. Hamilton, *The heat equation shrinking convex plane curves*, J. Diff. Geom. **23** (1986), 69–96.

[12] M.A. Grayson, *The heat equation shrinks embedded plane curves to points*, J. Diff. Geom. **26** (1987), 285–314.

[13] Richard S. Hamilton, *Harnack estimate for the mean curvature flow*, J. Differential Geom. **41** (1995), no. 1, 215–226.

[14] ———, *Isoperimetric estimates for the curve shrinking flow in the plane*, Modern methods in complex analysis (Princeton, NJ, 1992), Ann. of Math. Stud., vol. 137, Princeton Univ. Press, Princeton, NJ, 1995, pp. 201–222.

[15] Gerhard Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Diff. Geom. **20** (1984), 237–266.

[16] ———, *Asymptotic behavior for singularities of the mean curvature flow*, J. Diff. Geom. **31** (1990), 285–299.

[17] ———, *Local and global behaviour of hypersurfaces moving by mean curvature*, Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990), Proc. Sympos. Pure Math., vol. 54, Amer. Math. Soc., Providence, RI, 1993, pp. 175–191.

[18] ———, *A distance comparison principle for evolving curves*, Asian J. Math. **2** (1998), no. 1, 127–133.

[19] Gerhard Huisken and Alexander Polden, *Geometric evolution equations for hypersurfaces*, Calculus of variations and geometric evolution problems (Ce-

traro, 1996), Lecture Notes in Math., vol. 1713, Springer, Berlin, 1999, pp. 45–84.

[20] Brian White, *A local regularity theorem for mean curvature flow*, Ann. of Math. (2) **161** (2005), no. 3, 1487–1519.