The Logarithmic Sobolev Inequality: Stability, Instability, and Improved Convergence Rates for the Ornstein-Uhlenbeck Flow

Nikita Simonov

Gradient Flows Face to Face in Graná

IMAG, Granada, September 19, 2025





References and collaborators

Stability for the Logarithmic Sobolev Inequality.

G. Brigati, J. Dolbeault, and N.S., Journal of Functional Analysis (2024)

Logarithmic Sobolev inequality: a review on stability and instability results.

G. Brigati, J. Dolbeault, and N.S., `https://arxiv.org/pdf/2504.08658`

Outline of the talk:

Part I: ***The Logarithmic Sobolev Inequality and the Ornstein-Uhlenbeck Flow***

▷ *The Logarithmic Sobolev Inequality: Best Constant and Optimizers*

▷ *Rates of Convergence and the stability Issue: a Question Raised by Brezis and Lieb*

▷ *Positive and Negative Results*

Part II: ***A Constructive Stability Result Based on the Ornstein-Uhlenbeck Flow***

▷ *A Stability Result for the Logarithmic Sobolev Inequality*

▷ *Ideas and Strategy of the Proof*

Part I: The Logarithmic Sobolev Inequality and The Ornstein-Uhlenbeck Flow

The Logarithmic Sobolev Inequality, Historical Facts and Consequences

Let $d \geq 1$, the *Logarithmic Sobolev Inequality* states that, using $\|f\|_{p,\gamma} = \left(\int_{\mathbb{R}^d} |f|^p \, d\gamma\right)^{\frac{1}{p}}$,

$$\int_{\mathbb{R}^d} |\nabla f|^2 \, d\gamma \geq \frac{1}{2} \int_{\mathbb{R}^d} |f|^2 \log |f|^2 \, d\gamma \quad \text{for any } f \in \mathrm{H}^1(\mathbb{R}^d, d\gamma) \quad \text{such that } \|f\|_{2,\gamma} = 1 \,. \quad \text{(A)}$$

Where the measure $d\gamma = \gamma(x)dx$ is $\gamma(x) = (2\pi)^{-\frac{d}{2}} \, e^{-\frac{|x|^2}{2}}$.

▷ For $f \geq 0$, by considering the change of variable $f = \sqrt{u}$ so that $2\,u\,\nabla f = \nabla u$:

$$\mathcal{I}_\gamma(u) := \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{u} \, d\gamma \geq 2 \int_{\mathbb{R}^d} u \log u \, d\gamma =: \mathcal{E}_\gamma(u) \quad \text{with} \quad \|u\|_{1,\gamma} = 1 \,. \quad \text{(B)}$$

▷ Originally introduced by L. Gross (1975) and P. Federbush (1969) [15 proofs by M. Ledoux]

▷ There is an Euclidean version of (A) , apply $w = f\sqrt{\gamma}$:

$$\int_{\mathbb{R}^d} |\nabla w|^2 \, dx \geq \frac{1}{2} \int_{\mathbb{R}^d} w^2 \log w^2 \, dx + \frac{d}{4} \log(2\pi e^2) \quad \text{where} \quad \int_{\mathbb{R}^d} w^2 \, dx = 1 \,. \quad \text{(C)}$$

▷ Consider in (C) $w_\lambda(x) = \lambda^{\frac{d}{4}} w(\sqrt{\lambda} x)$: optimize in λ :

$$\int_{\mathbb{R}^d} |\nabla w|^2 \, dx \geq \frac{\pi \, d \, e}{2} \exp\left(\frac{2}{d} \int_{\mathbb{R}^d} w^2 \log w^2 \, dx\right) \quad \text{(D)}$$

Euclidean Logarithmic Sobolev Inequality in invariant form: A. Stam (1959), F. Weissler (1978)

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▷ Convergence to equilibrium for the Ornstein-Uhlenbeck flow

$$\frac{\partial h}{\partial t} = \Delta h - x \cdot \nabla h \, (t,x) \in (0,\infty) \times \mathbb{R}^d \quad \text{with} \quad h(t=0,x) = h_0$$

Consider the entropy $\mathcal{E}_\gamma(h(t))$, therefore $\dfrac{d\mathcal{E}_\gamma(h)}{dt} = -\mathcal{I}_\gamma(h)$. We obtain

$$\mathcal{E}_\gamma(h(t)) \leq \mathcal{E}_\gamma(h_0) \, e^{-2\,t} \quad \text{which implies, for instance} \quad \|h(t) - 1\|_{2,\gamma} \leq C \, e^{-2\,t} \,.$$

Q: is the rate sharp ?

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The Rate is Sharp: Optimality and Invariance

▷ Carlen (1991) proved that the the constant $\frac{1}{2}$ in the *Logarithmic Sobolev Inequality* is *optimal*:

$$\int_{\mathbb{R}^d} |\nabla f|^2 \, d\gamma - \frac{1}{2} \int_{\mathbb{R}^d} |f|^2 \log |f|^2 \, d\gamma = 0 \quad \text{iff} \quad f \in \mathcal{M} = \{ e^{\frac{b\cdot x}{2} - \frac{|b|^2}{4}} \, : \, b \in \mathbb{R}^d \} \tag{A}$$

or, equivalently, for $f \geq 0$ after the change of variables $f = \sqrt{u}$ that

$$\mathcal{I}_\gamma(u) - \frac{1}{2}\mathcal{E}_\gamma(u) = \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{u} \, d\gamma - \frac{1}{2} \int_{\mathbb{R}^d} u \log u \, d\gamma = 0 \quad \textit{iff} \quad u \in \mathcal{M}_\star = \{ e^{b\cdot x - \frac{|b|^2}{2}} \, : \, b \in \mathbb{R}^d \} \tag{B}$$

Recall that the assumption is $\|f\|_{2,\gamma} = \|u\|_{1,\gamma} = 1$.

▷ The Ornstein-Uhlenbeck flow leaves $\mathcal{M}_\star$ invariant: for $\partial_t h = \mathcal{L} h$, where $\mathcal{L} = \Delta - x \cdot \nabla$, we have

$$\text{if } h_0 \in \mathcal{M}_\star \quad \text{then} \quad \forall t > 0 \, , \, h(t) \in \mathcal{M}_\star \text{ and } \mathcal{E}_\gamma(h(t)) = e^{-2\,t} \mathcal{E}_\gamma(h_0) \, .$$

The Quest for Improvements: Rates and Inequality

$$\frac{\partial h}{\partial t} = \mathcal{L}h = \Delta h - x \cdot \nabla h$$

Ornstein-Uhlenbeck flow

Optimal Constant in LSI

$$\int_{\mathbb{R}^d} |\nabla f|^2 \, d\gamma \geq \frac{1}{2} \int_{\mathbb{R}^d} |f|^2 \log |f|^2 \, d\gamma$$

equality in $\mathcal{M} = \{e^{\frac{b\cdot x}{2} - \frac{|b|^2}{4}} : b \in \mathbb{R}^d\}$

$\longleftrightarrow$

Optimal Rates of convergence

$$\mathcal{E}_\gamma(h(t)) \leq e^{-2t} \mathcal{E}_\gamma(h_0)$$

leaves invariant $\mathcal{M}_\star = \{e^{b\cdot x - \frac{|b|^2}{2}} : b \in \mathbb{R}^d\}$

Q: what kind of improvements on the left imply improvements on the right and vice-versa?

Improved Rates Give an Better Constant

Consider again the Ornstein-Uhlenbeck flow

$$\frac{\partial h}{\partial t} = \Delta h - x \cdot \nabla h \,(t,x) \in (0,\infty) \times \mathbb{R}^d \quad \text{with} \quad h(t=0,x) = h_0 \quad \text{and} \quad \|h_0\|_{1,\gamma} = 1$$

What happens if, for some $\delta > 0$ and some solution h, we have

$$\mathcal{E}_\gamma(h(t)) \le e^{-(2+\delta)\,t} \mathcal{E}_\gamma(h_0) \quad \forall t > 0 \quad ?$$

We find that

$$\frac{\mathcal{E}_\gamma(h_0) - \mathcal{E}_\gamma(h(t))}{t} = \frac{1}{t} \int_0^t \mathcal{I}_\gamma(h(\tau)) d\tau \ge \mathcal{E}_\gamma(h_0) \, \frac{1 - e^{-(2+\delta)\,t}}{t} \, .$$

By taking the limit $t \to 0$ we obtain

$$\mathcal{I}_\gamma(h_0) \ \ge (2+\delta)\, \mathcal{E}_\gamma(h_0)$$

Key Idea: **Improved rates** along the flow give **Improved Constant in LSI** for the initial datum.

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Key Idea: **Improved rates** along the flow give **Improved Constant in LSI** for the initial datum.

Improved Rates Give an Better Constant: Consequence

Let us consider again the inqeuality (recall $\|h_0\|_{1,\gamma} = 1$)

$$\mathcal{I}_\gamma(h_0) \geq (2+\delta)\,\mathcal{E}_\gamma(h_0)\,,$$

Define $f_0^2 = h_0$ and recall that $\nabla h_0 = 2 f_0 \nabla f_0$, this gives (recall that $\|f_0\|_{2,\gamma} = 1$):

$$\frac{4}{2+\delta}\int_{\mathbb{R}^d} |\nabla f_0|^2\, d\gamma \geq \int_{\mathbb{R}^d} f_0^2 \log f_0^2\, d\gamma$$

or alternatively

$$2\int_{\mathbb{R}^d} |\nabla f_0|^2\, d\gamma - \int_{\mathbb{R}^d} f_0^2 \log f_0^2\, d\gamma \geq \frac{2\,\delta}{2+\delta}\int_{\mathbb{R}^d} |\nabla f_0|^2\, d\gamma = \frac{2\,\delta}{2+\delta}\int_{\mathbb{R}^d} |\nabla f_0 - \nabla(1)|^2\, d\gamma$$

This last inequality implies the following one which opens the gates towards the *stability problem*

$$2\int_{\mathbb{R}^d} |\nabla f_0|^2\, d\gamma - \int_{\mathbb{R}^d} f_0^2 \log f_0^2\, d\gamma \geq \mathsf{c} \inf_{g\in\mathcal{M}} \int_{\mathbb{R}^d} |\nabla f_0 - \nabla g|^2\, d\gamma\,,$$

where

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Stability: a question raised by Brezis and Lieb (1985)

Is there a natural way to bound from below

$$\delta_{LSI}[f] = 2\int_{\mathbb{R}^d} |\nabla f|^2 \, d\gamma - \int_{\mathbb{R}^d} f^2 \log f^2 \, d\gamma \geq 0$$

in terms of a "distance" to the manifold of the optimal functions?
(Similar questions in probability go back to P. Levy (1936) for the Young's inequality)

▷ Originally the question was asked for the Sobolev's inequality, $d \geq 3$

$$\int_{\mathbb{R}^d} |\nabla g|^2 \, dx - \left(\int_{\mathbb{R}^d} |g|^{\frac{2d}{d-2}} \, dx \right)^{\frac{d-2}{d}} \geq 0$$

▷ We are looking for an inequality of the type, for some $\beta > 0$:

$$\delta_{LSI}[f] \geq \mathsf{c}\, d(f, \mathcal{M})^\beta = \inf_{g \in \mathcal{M}} d(f,g)^\beta \quad \text{or the stronger} \quad \delta_{LSI}[f] \geq \mathsf{c}\, d(f,1)^\beta$$

A (non-exhaustive) list for the *distance*

- ▷ $d(f,g) = \int_{\mathbb{R}^d} |\nabla f - \nabla g|^2 \, d\gamma$ or the $\dot{\mathrm{H}}^1_\gamma(\mathbb{R}^d)$ distance.
- ▷ $d(f,g) = \|f - g\|_{p,\gamma}$, $1 < p < \infty$
- ▷ $d(f,g) = \mathcal{W}_p(|f|^2\gamma, |g|^2\gamma)$ or the Wasserstein distance.

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A Non-Exhaustive List of Results

A more exhaustive list can be found in Brigati-Dolbeault-NS (2025).

▷ Results for the (A) or (B) form: $\delta_{LSI-A} = \int_{\mathbb{R}^d} |\nabla f|^2 \, d\gamma \geq \frac{1}{2} \int_{\mathbb{R}^d} |f|^2 \log |f|^2 \, d\gamma$:

- - Carlen (1991): $\delta_{LSI-A}(w) \geq \int_{\mathbb{R}^d} |\mathcal{T} w|^2 \log |\mathcal{T} w|^2 \, d\gamma$ where $\mathcal{T}$ is the *Wiener transform*

- - Bobkov, Gozlan, Roberto, Samson (2014) $\delta_{LSI-A}(f) \geq C \, \mathcal{W}_2^4(f^2 d\gamma, d\gamma)$ if $\int_{\mathbb{R}^d} f^2 |x|^2 \, d\gamma \leq d$

- - Fathi-Indrei-Ledoux (2016) $\delta_{LSI-A}(f) \geq \sigma \int_{\mathbb{R}^d} |\nabla f|^2 \, d\gamma$
if $d\mu = f \, d\gamma$ satisfies a *Poincaré inequality*

- - Indrei-Kim (2019): $\delta_{LSI-A}(f) \geq C \, \|f - 1\|_{1,\gamma}^4$ if f is centered and $\int_{\mathbb{R}^d} |x|^2 f^2 \, d\gamma \leq M$

- - Dolbeault-Esteban-Figalli-Frank-Loss $\delta_{LSI-A}(f) \geq C \int_{g \in \mathcal{M}} \|f - g\|_{2,\gamma}$ no assumption on f

- - Indrei (2025): $\delta_{LSI-A}^{\frac{3}{4}}(f) + \delta_{LSI-A}^{\frac{1}{8}}(f) \geq C \|f - 1\|_{W^{1,1}(d\gamma)}$ if $\int_{\mathbb{R}^d} f^2 |x|^2 \, d\gamma \leq M$

▷ A few results are available also for the (C) or (D) form:

$$\delta_{LSI-C}(w) \int_{\mathbb{R}^d} |\nabla w|^2 \, dx \geq \frac{1}{2} \int_{\mathbb{R}^d} w^2 \log w^2 \, dx + \frac{d}{4} \log(2\pi e^2)$$

Dolbeault-Toscani (2016) and Feo-Posteraro-Indrei-Roberto

What about the wanted inequalities without moment assumptions:

$$\delta_{LSI}[f] \geq \mathsf{c} \, d(f, \mathcal{M}) = \inf_{g \in \mathcal{M}} \int_{\mathbb{R}^d} |\nabla f - \nabla g|^2 \, d\gamma \text{ or the stronger } \delta_{LSI}[f] \geq \mathsf{c} \int_{\mathbb{R}^d} |\nabla f|^2 \, d\gamma = d(f, 1)?$$

A Non-Exhaustive List of Results

A more exhaustive list can be found in Brigati-Dolbeault-NS (2025).

▷ Results for the (A) or (B) form: $\delta_{LSI-A} = \int_{\mathbb{R}^d} |\nabla f|^2 \, d\gamma \geq \frac{1}{2} \int_{\mathbb{R}^d} |f|^2 \log |f|^2 \, d\gamma$:

- - Carlen (1991): $\delta_{LSI-A}(w) \geq \int_{\mathbb{R}^d} |\mathcal{T} w|^2 \log |\mathcal{T} w|^2 \, d\gamma$ where $\mathcal{T}$ is the *Wiener transform*

- - Bobkov, Gozlan, Roberto, Samson (2014) $\delta_{LSI-A}(f) \geq C \, \mathcal{W}_2^4(f^2 d\gamma, d\gamma)$ if $\int_{\mathbb{R}^d} f^2 |x|^2 \, d\gamma \leq d$

- - Fathi-Indrei-Ledoux (2016) $\delta_{LSI-A}(f) \geq \sigma \int_{\mathbb{R}^d} |\nabla f|^2 \, d\gamma$
if $d\mu = f \, d\gamma$ satisfies a *Poincaré inequality*

- - Indrei-Kim (2019): $\delta_{LSI-A}(f) \geq C \, \|f - 1\|_{1,\gamma}^4$ if f is centered and $\int_{\mathbb{R}^d} |x|^2 f^2 \, d\gamma \leq M$

- - Dolbeault-Esteban-Figalli-Frank-Loss $\delta_{LSI-A}(f) \geq C \int_{g \in \mathcal{M}} \|f - g\|_{2,\gamma}$ no assumption on f

- - Indrei (2025): $\delta_{LSI-A}^{\frac{3}{4}}(f) + \delta_{LSI-A}^{\frac{1}{8}}(f) \geq C \|f - 1\|_{W^{1,1}(d\gamma)}$ if $\int_{\mathbb{R}^d} f^2 |x|^2 \, d\gamma \leq M$

▷ A few results are available also for the (C) or (D) form:

$$\delta_{LSI-C}(w) \int_{\mathbb{R}^d} |\nabla w|^2 \, dx \geq \frac{1}{2} \int_{\mathbb{R}^d} w^2 \log w^2 \, dx + \frac{d}{4} \log(2\pi e^2)$$

Dolbeault-Toscani (2016) and Feo-Posteraro-Indrei-Roberto

What about the wanted inequalities without moment assumptions:

$\delta_{LSI}[f] \geq \mathsf{c} \, d(f, \mathcal{M}) = \inf_{g \in \mathcal{M}} \int_{\mathbb{R}^d} |\nabla f - \nabla g|^2 \, d\gamma$ or the stronger $\delta_{LSI}[f] \geq \mathsf{c} \int_{\mathbb{R}^d} |\nabla f|^2 \, d\gamma = d(f, 1)$?

Counterexamples to Stability

▷ Indrei - Marcon (2014)) pointed out the role of moments in *instability examples*.
▷ In Kim (2018), Indrei-Kim (2021 and 2022) and Eldan, Lehec and Shenfeld (2020) the following example of sequence is constructed: let $n \in \mathbb{N}$ and in dimension $d = 1$:

$$g_n := \begin{cases} 1 & \text{if} \quad |x| \leq \frac{n}{2} - \frac{1}{2n} \\ \psi_n(|x|) & \text{if} \quad \frac{n}{2} - \frac{1}{2n} \leq |x| \leq \frac{n}{2} \\ \sqrt{\varepsilon_n}\, \exp\left(\frac{n}{2}|x| - \frac{n^2}{4}\right) & \text{if} \quad |x| \geq \frac{n}{2}\,. \end{cases}$$

where $\varepsilon_n \sim n^{-2}$; consider in particular $f_n := g_n / \|g_n\|_{2,\gamma}$.
▷ We have, in particular, as $n \to \infty$

$$\int_{\mathbb{R}^d} |\nabla f_n|^2 \, d\gamma - \frac{1}{2} \int_{\mathbb{R}^d} f_n^2 \log f_n^2 \, d\gamma \to 0 \quad \text{and} \quad \liminf_{n\to\infty} \int_{\mathbb{R}^d} |\nabla f_n|^2 \, d\gamma \geq 0\,.$$

In Brigati-Dolbeault-NS(2025) we also have shown:

$$\liminf_{n\to\infty} \inf_{g\in\mathcal{M}} \|\nabla f_n - \nabla g\|_{2,\gamma} \geq 0 \quad \text{where} \quad \mathcal{M} = \{e^{\frac{b\cdot x}{2} - \frac{|b|^2}{4}} : b \in \mathbb{R}^d\}$$

[0] We thank M. Fathi for pointing out to us the above results

Counterexamples to Stability: an Explanation on What is Happening

Recall the change of variable used in Carlen (1991): from

$$f = \frac{w}{\sqrt{\gamma}} \qquad \textit{sothat } \nabla f = \frac{\nabla w}{\sqrt{\gamma}} - \frac{w}{2}\gamma^{-\frac{3}{2}}\nabla\gamma = \frac{1}{\sqrt{\gamma}}\left(\nabla w + \frac{x}{2}w\right)$$

so that

$$\int_{\mathbb{R}^d} |\nabla f|^2 \, d\gamma = \int_{\mathbb{R}^d} \left(|\nabla w|^2 + \frac{|x|^2}{4}w^2 - \frac{d}{2}w^2\right) dx$$

and

$$\frac{1}{2}\int_{\mathbb{R}^d} f^2 \log f^2 \, d\gamma = \frac{1}{2}\int_{\mathbb{R}^d} w^2 \log w^2 \, dx + \frac{1}{4}\int_{\mathbb{R}^d} |x|^2 w^2 \, dx + \frac{d}{4}\log(2\pi)\int_{\mathbb{R}^d} w^2 \, dx\,.$$

Therefore

$$\int_{\mathbb{R}^d} |\nabla f|^2 \, d\gamma - \frac{1}{2}\int_{\mathbb{R}^d} f^2 \log f^2 \, d\gamma = \cdots \text{no terms in } \int_{\mathbb{R}^d} |x|^2 w^2 \, dx\,,$$

but for the previous sequence f_n :

$$\int_{\mathbb{R}^d} |\nabla f_n|^2 \, d\gamma \geq \frac{1}{4}\int_{\mathbb{R}^d} |x|^2 f_n^2 \, d\gamma \quad \text{and} \quad \liminf_{n\to\infty} \inf_{g\in\mathcal{M}} \|\nabla f_n - \nabla g\|_{2,\gamma} \geq \liminf_{n\to\infty} \int_{\mathbb{R}^d} |x|^2 f_n^2 \, d\gamma$$

Key Idea: No stability estimates without a control of second moments!

Q: what can we do if we do have a control of tails?

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Part II: A Constructive Stability Result Based on the Ornstein-Uhlenbeck Flow

A Constructive Stability Result

Consider the space:

$$\mathcal{H} := \left\{ f \in \mathrm{H}^1(\mathbb{R}^d, d\gamma) \ : \ \|f\|_{2,\gamma} = 1 \right\}$$

where $\|f\|_{2,\gamma} = \int_{\mathbb{R}^d} f^2 d\gamma$ and $\gamma(x) = (2\pi)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2}}$

Theorem (G. Brigati, J. Dolbeault, N.S. (2024))

Let $d \geq 1$. For any $\varepsilon > 0$ and $C > 0$, there is an explicit $\eta \in (0,1)$ such that

$$(1-\eta) \, \|\nabla f\|_{2,\gamma}^2 \geq \frac{1}{2} \int_{\mathbb{R}^d} |f|^2 \, \log |f|^2 \, d\gamma$$

for any $u \in \mathcal{H}$ such that

$$\int_{\mathbb{R}^d} x \, |f|^2 \, d\gamma = 0 \quad \textit{and} \quad \iint_{\mathbb{R}^d \times \mathbb{R}^d} |f(x)|^2 \, |f(y)|^2 \, e^{\varepsilon \, |x-y|^2} d\gamma(x) \, d\gamma(y) \leq C \, . \tag{5.1}$$

The proof is based

$$\frac{\partial h}{\partial t} = \mathcal{L} h \, , \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d \, , \tag{5.2}$$

where $\mathcal{L} h := \Delta h - x \cdot \nabla h$ denotes the *Ornstein–Uhlenbeck* operator and $h = f^2$

Strategy of the Proof

▷ Recall first that the inequality

$$\int_{\mathbb{R}^d} |\nabla f|^2 \, d\gamma \geq \frac{1}{2} \int_{\mathbb{R}^d} |f|^2 \log |f|^2 \, d\gamma \quad \text{for any } f \in \mathrm{H}^1(\mathbb{R}^d, d\gamma) \quad \text{such that } \|f\|_{2,\gamma} = 1 \, .$$

can be written as:

$$\mathcal{I}_\gamma(u) := \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{u} \, d\gamma \geq 2 \int_{\mathbb{R}^d} u \log u \, d\gamma =: \mathcal{E}_\gamma(u) \quad \text{with} \quad \|u\|_{1,\gamma} = 1 \, .$$

when $f \geq 0$, by considering the change of variable $f = \sqrt{u}$ so that $2\, u\, \nabla f = \nabla u$.
▷ Recall at the same time that for the Ornstein-Uhlenbeck flow

$$\frac{\partial h}{\partial t} = \Delta h - x \cdot \nabla h \, (t, x) \in (0, \infty) \times \mathbb{R}^d \quad \text{with} \quad h(t = 0, x) = h_0$$

Consider the entropy $\mathcal{E}_\gamma(h(t))$, therefore $\dfrac{d\mathcal{E}_\gamma(h)}{dt} = -\mathcal{I}_\gamma(h)$. We obtain

$$\mathcal{E}_\gamma(h(t)) \leq \mathcal{E}_\gamma(h_0)\, e^{-2\,t} \, .$$

Strategy: Obtain an **improved rate along the flow**

Intuition: Log-Concavity

▷ A function $\phi : \mathbb{R}^d \to \mathbb{R}$ is said to be Log-Concave if $\log \phi$ is Concave:

$$\phi(x) = e^{-V(x)} \quad \text{for} \quad V : \mathbb{R}^d \to \mathbb{R} \quad \text{Convex} .$$

▷ Brascamp-Lieb (1976) : log-concavity is *preserved* along the Ornstein-Uhlenbeck flow.
▷ It is known that log-concave solutions converge rapidly Fathi-Indrei-Ledoux (2014)

Q: is log-concavity created along the Ornstein-Uhlenbeck flow ?

Unfortunately no: Brigati-Pedrotti (2024)

▷ However, what can be "created" is the following: solutions to the Ornstein-Uhlenbeck equation may satisfy a Poincaré inequality Chen, Chewi, and Niles-Weed (2021)

$$\lambda_\mu \int_{\mathbb{R}^d} |\phi - \overline{\phi}|^2 d\mu \leq \int_{\mathbb{R}^d} |\nabla \phi|^2 \, d\mu \quad \text{where} \quad \overline{\phi} = \int_{\mathbb{R}^d} \phi d\mu \, , \tag{P}$$

and μ is a probability measure . Consider

$$\frac{\partial h}{\partial t} = \Delta h - x \cdot \nabla h \, (t, x) \in (0, \infty) \times \mathbb{R}^d \quad \text{with} \quad h(t = 0, x) = h_0 \geq 0 \quad \text{such that}$$

$$\iint_{\mathbb{R}^d} h_0(x) h_0(y) e^{\varepsilon |x-y|^2} d\gamma(x) d\gamma(y) \leq C \text{ then } \mu(t) = h(t) d\gamma \quad \text{satisfies (P)} \quad \forall t \geq \log(1 + \frac{1}{\varepsilon})$$

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The (Simplified) Final Strategy

Consider

$$\frac{\partial h}{\partial t} = \Delta h - x \cdot \nabla h \,(t,x) \in (0,\infty) \times \mathbb{R}^d \quad \text{with} \quad h(t=0,x) = h_0 \geq 0$$

i) Assume that

$$\iint_{\mathbb{R}^d} h_0(x) h_0(y) e^{\varepsilon |x-y|^2} d\gamma(x) d\gamma(y) \leq C$$

ii) By using Chen, Chewi, and Niles-Weed (2021) we get that

$$\mu(t) = h(t) d\gamma \quad satisfies(P) \quad \forall t \geq t_\varepsilon = \log(1 + \frac{1}{\varepsilon})$$

iii) By using Fathi-Indrei-Ledoux (2014) the above property implies that

$$\mathcal{E}_\gamma(h(t)) \leq \mathcal{E}_\gamma(h(t_\varepsilon))\, e^{-(2+\delta(t))t} \text{ or equivalently } 2(1-\eta(t)) \int_{\mathbb{R}^d} \frac{|\nabla h(t)|^2}{h(t)} d\gamma \geq \int_{\mathbb{R}^d} h(t) \log h(t)\, d\gamma$$

iv) Consider the quotient $\mathcal{Q}(t) = \mathcal{I}_\gamma(h(t))/\mathcal{E}_\gamma(h(t))$. By the Bakry-Emery method (1985) we know that

$$-\frac{d^2 \mathcal{E}_\gamma(h(t))}{dt^2} = \frac{d\mathcal{I}_\gamma(h(t))}{dt} \leq -2\mathcal{I}_\gamma(h(t)) \quad \text{which implies that} \quad \frac{dQ(t)}{dt} \leq 2Q(t)(2Q(t)-1)$$

Thank you for your attention!