On a Cross-Diffusion System with Independent Drifts: The Existence of Totally Mixed Solutions [1]

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[1] Based on joint work with Alpár R. Mészáros at Durham University

Motivation

The cross-diffusion system

$$\begin{cases} \partial_t \rho_i + \partial_x(\rho_i \boldsymbol{v}_i) = 0, \\ \boldsymbol{v}_i = -\partial_x P, \quad P = \Phi'(\rho_1 + \rho_2), \end{cases} \qquad \text{for } i \in \{1, 2\}, \tag{1}$$

introduced in[2], describes the evolution of two interacting species whose agents only disperse in response to overall population density.

[2] Stavros N. Busenberg and Curtis C. Travis (1983). "Epidemic models with spatial spread due to population migration". In: *J. Math. Biol.* 16.2, pp. 181–198.

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introduced in[2], describes the evolution of two interacting species whose agents only disperse in response to overall population density.

System (1) has since become ubiquitous in the modelling of:

- Heterogeneous cell populations,
- Tumour growth,
- Interacting migratory species,
- Epidemiology.

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Motivation

Question: Can we incorporate additional convective effects?

$$\begin{cases} \partial_t \rho_i + \partial_x(\rho_i \boldsymbol{v}_i) = 0, \\ \boldsymbol{v}_i = -\partial_x(P + V_i), \quad P = \Phi'(\rho_1 + \rho_2), \end{cases} \qquad \text{for } i \in \{1, 2\}. \tag{1}$$

Features: No Drift

System (1) exhibits 2-Wasserstein gradient flow structure, associated with the energy:

$$\mathcal{E}[\rho_1, \rho_2] := \int \Phi(\rho_1 + \rho_2)\, \mathrm{d}x$$

The existence and uniqueness of weak solutions is well-established[2].

Solutions can form sharp inter-species interfaces which correspond to jump discontinuities in each density.

[2] M. Bertsch, M. E. Gurtin, and D. Hilhorst (1987). "On interacting populations that disperse to avoid crowding: the case of equal dispersal velocities". In: *Nonlinear Anal.* 11.4, pp. 493–499.

Features: With Drift

In the presence of drift terms, the associated energy becomes:

$$\mathcal{F}[\rho_1, \rho_2] := \int \Phi(\rho_1 + \rho_2) + V_1\rho_1 + V_2\rho_2 \, \mathrm{d}x$$

The existence of solutions has so far only been proven under restrictive assumptions.

The Problem with Existence

The main obstacle in establishing solutions for System (1) is posed by the product term:

$$\rho_i \partial_x \Phi'(\rho_1 + \rho_2). \tag{2}$$

In particular, if we are given a suitable approximation of System (1), then it is not immediately clear that the limit of the product term (2) can be identified with the product of its limits.

A New Approach

We consider System (1) equipped with a logarithmic pressure law and periodic boundary conditions. In this instance, the system reduces to

$$\partial_t \rho_i = \partial_x(\rho_i \partial_x(\log(\rho_1 + \rho_2) + V_i)) \text{ for } i \in \{1, 2\}, \quad \text{on } (0, T) \times \mathbb{T}. \tag{3}$$

For this system, we are able to show that BV regularity for the logarithmic ratio

$$\log\left(\frac{\rho_1}{\rho_2}\right)$$

is propagated across time.

Such regularity allows the derivation of BV estimates for the individual species densities from which it is possible to recover their strong compactness.

Theorem[2]

(**H1**) $\mathcal{F}[\rho_{1,0}, \rho_{2,0}] < +\infty$;

(**H2**) $\log\left(\frac{\rho_{1,0}}{\rho_{2,0}}\right) \in BV(\mathbb{T})$;

(**H3**) $V_1, V_2 \in W^{2,1}(\mathbb{T})$;

(**H4**) $V_1 - V_2 \in W^{3,1}(\mathbb{T})$.

Theorem

If Hypotheses (**H1**)-(**H4**) *hold, then there exists a weak solution* (ρ_1, ρ_2) *to System* (3) *with potentials* V_1, V_2 *and initial data* $(\rho_{1,0}, \rho_{2,0})$. *Moreover, this solution possesses the regularity:*

$$\rho_1, \rho_2 \in L^2([0, T]; BV(\mathbb{T})).$$

In particular, the above hypotheses allow for differing individual drifts in the presence of mixed initial densities.

Alpár R. Mészáros and G. P. (2025). *On a Cross-Diffusion System with Independent Drifts and no Self-Diffusion: The Existence of Totally Mixed Solutions*. arXiv: 2504.18484.

Interface Properties: Total Mixing

Our approach shows that BV regularity for

$$\log\left(\frac{\rho_1}{\rho_2}\right)$$

is propagated across time. Moreover, since

$$BV(\mathbb{T}) \hookrightarrow L^{\infty}(\mathbb{T}),$$

ρ_1 and ρ_2 must share the same support and approach points of blow up and vacuum at a similar asymptotic rate to one another.

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We refer to densities satisfying this property as being *totally mixed*.

In this instance, species can not form complete interfaces in finite time.

Final Thoughts

New estimates provide:

- BV regularity,
- a weak existence theory,
- qualitative understanding of interface formation.

There are still lots of exciting directions to be explored. For instance:

- the (non)-uniqueness of solutions.

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Thank you for listening!