Nonlinear diffusions as limits of kinetic equations

Daniel Morris

collaboration with Havva Yoldaş, Josephine Evans

Gradient flows face-to-face 5 in Granada

Background

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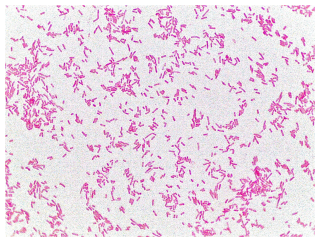
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E.g. chemotaxis of E. coli bacteria can be modelled by a,

velocity-jump process → run and tumble → Keller-Segel





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We study the *diffusive scaling* limit of a kinetic equation,

large time scale, small mean free path regime

$(t, x, v) \mapsto (\varepsilon^2 t, \varepsilon x, v)$, where ε is the *Knudsen number* — the ratio of the mean free path to a characteristic length scale.

The models

Nonlinear kinetic equation

We introduce the following kinetic equation,

$$\begin{aligned} \varepsilon \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon &= \frac{1}{\varepsilon} \rho_{f_\varepsilon}^\alpha (\rho_{f_\varepsilon} \mathcal{M} - f_\varepsilon), \\ f_\varepsilon|_{t=0} &= f_{\varepsilon,\mathrm{in}}, \end{aligned} \tag{1}$$

where $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$, $\rho_f := \int f \, \mathrm{d}v$, $\mathcal{M}(v) := (2\pi)^{-\frac{d}{2}} e^{-\frac{|v|^2}{2}}$, and $\alpha \in \mathbb{R}$.

Velocity-jump process with density-dependent turning rate.

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Velocity-jump process with density-dependent turning rate.

In the $\varepsilon \to 0$ limit, by a formal Hilbert series expansion, we expect $f_\varepsilon \to \rho \mathcal{M}$ where ρ solves a *nonlinear diffusion* equation.

Nonlinear diffusion equation

$$\begin{aligned} \partial_t \rho &= \nabla_x \cdot \left(\rho^{-\alpha} \nabla_x \rho \right), \\ \rho\big|_{t=0} &= \rho_{\text{in}}. \end{aligned} \tag{2}$$

$\alpha < 0$: porous medium equation.

$\alpha = 0$: heat equation.

$\alpha > 0$: fast diffusion equation.

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In this work, we show that $f_\varepsilon \to \rho\mathcal{M}$ strongly in $L^2(\mathcal{M}^{-1}\,\mathrm{d}x\,\mathrm{d}v)$, uniformly in time, with a rate, for any $\alpha \in \mathbb{R}$.

Results

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Theorem 1 [*Well-posedness, propagation of bounds*]

Let $A > 1, \alpha \in \mathbb{R}$, and let $f_{\varepsilon,\text{in}} \in L^1(\mathbb{T}^d \times \mathbb{R}^d)$ satisfy $A^{-1}\mathcal{M} \leq f_{\varepsilon,\text{in}} \leq A\mathcal{M}$. Then (1) admits a unique weak solution $f_\varepsilon \in C([0,\infty); L^1(\mathbb{T}^d \times \mathbb{R}^d))$ and f_ε satisfies $A^{-1}\mathcal{M} \leq f_\varepsilon \leq A\mathcal{M}$.

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Sketch of proof

- Solve for f as a fixed point of the map $\Gamma[f] = g$ where g solves

$$\partial_t g + \nabla_x \cdot g = \rho_f^\alpha \left(\rho_f \mathcal{M} - g\right),$$
$$g(t=0) = f_{\text{in}},$$

  in $\mathcal{V} := \{\phi \in C([0,T]; L^1(\mathbb{T}^d \times \mathbb{R}^d)) : A^{-1}\mathcal{M} \leq \phi \leq A\mathcal{M}\}$.
- Maximum principle $\rightarrow$ propagation of Maxwellian bounds.

Theorem 2 [*Long-time behaviour*]

Let $A > 1, \alpha \in \mathbb{R}, f_{\varepsilon,\text{in}} \in L^1(\mathbb{T}^d \times \mathbb{R}^d)$ satisfy $A^{-1}\mathcal{M} \leq f_{\varepsilon,\text{in}} \leq A\mathcal{M}$ and $\|f_{\varepsilon,\text{in}}\|_{L^1} = 1$, and let f_ε be the corresponding solution of (1). Then there exists $\varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$ then

$$\|f_\varepsilon - \mathcal{M}\|^2_{L^2_{x,v}(\mathcal{M}^{-1})} \leq 3e^{-\gamma t}\|f_{\varepsilon,\text{in}} - \mathcal{M}\|^2_{L^2_{x,v}(\mathcal{M}^{-1})}, \tag{3}$$

where $\gamma > 0$ is a constant depending only on α, A.

[1] J. Dolbeault, C. Mouhot, and C. Schmeiser (June 2015). "Hypocoercivity for linear kinetic equations conserving mass". In: *Trans. Amer. Math. Soc.* 367.6, pp. 3807–3828. ISSN: 0002-9947. DOI: 10.1090/S0002-9947-2015-06012-7

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Sketch of proof

- We adapt *linear* L^2 *hypocoercivity* methods[1] to our nonlinear equation.
- First show microscopic coercivity,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f_\varepsilon - \mathcal{M}\|^2_{L^2_{x,v}(\mathcal{M}^{-1})} \le -\frac{2}{\varepsilon^2 A^\alpha}\|(I - \Pi)f_\varepsilon\|^2_{L^2_{x,v}(\mathcal{M}^{-1})}, \quad \Pi f := \rho_f \mathcal{M},$$

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- Now add a macroscopic perturbation term,

$$I_\varepsilon := \int_{\mathbb{T}^d} j_\varepsilon \cdot (I - \Delta_x)^{-1} \nabla_x \rho_\varepsilon \, \mathrm{d}x,$$

and show there exist $\gamma, C > 0$, depending only on α, A, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|f_\varepsilon - \mathcal{M}\|^2_{L^2_{x,v}(\mathcal{M}^{-1})} + \varepsilon C I_\varepsilon\right) \leq -\gamma\left(\|f_\varepsilon - \mathcal{M}\|^2_{L^2_{x,v}(\mathcal{M}^{-1})} + \varepsilon C I_\varepsilon\right).$$

- Conclude with Grönwall and

$$\varepsilon C |I_\varepsilon| \leq \frac{1}{2} \|f_\varepsilon - \mathcal{M}\|^2_{L^2_{x,v}(\mathcal{M}^{-1})}.$$

Theorem 3 [*Diffusive asymptotics*]

Let $A > 1$ and $\varepsilon \in (0, \varepsilon_0)$ where ε_0 given by Theorem 2. Let $\{f_{\varepsilon,\text{in}}\} \subset L^1(\mathbb{T}^d \times \mathbb{R}^d)$ satisfy $\|f_{\varepsilon,\text{in}}\|_{L^1} = 1$ and $A^{-1}\mathcal{M} \leq f_{\varepsilon,\text{in}} \leq A\mathcal{M}$, and let f_ε be the corresponding solution of (1). Let $\rho_{\text{in}} \in C^2(\mathbb{T}^d)$ satisfy $\|\rho_{\text{in}}\|_{L^1} = 1$, $A^{-1} \leq \rho_{\text{in}} \leq A$, and let $\tilde{\rho}$ be the corresponding solution of (2). Suppose the initial data is well-prepared,

$$\|f_{\varepsilon,\text{in}} - \rho_{\text{in}}\mathcal{M}\|_{L^2_{x,v}(\mathcal{M}^{-1})} \leq \varepsilon,$$

then there exist constants $C, \kappa > 0$ depending only on $\alpha, A, \|\rho_{\text{in}}\|_{C^2}, d$ such that

$$\|f_\varepsilon - \tilde{\rho}\mathcal{M}\|_{L^\infty(\mathbb{R}_+; L^2_{x,v}(\mathcal{M}^{-1}))} \leq C\varepsilon^\kappa. \tag{4}$$

Sketch of proof

- Following Anceschi, Zhu[2] we study the asymptotics on short times and long times separately.

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Sketch of proof

- Following Anceschi, Zhu[2] we study the asymptotics on short times and long times separately.
- On long times we combine exponential relaxation to equilibrium of both solutions: for $t \geq T$,

$$\|f_\varepsilon - \tilde{\rho}\mathcal{M}\|_{L^2_{x,v}(\mathcal{M}^{-1})} \leq \|f_\varepsilon - \mathcal{M}\|_{L^2_{x,v}(\mathcal{M}^{-1})} + \|\tilde{\rho} - 1\|_{L^2_x} \lesssim e^{-\gamma T}.$$

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- On short times we write a Grönwall argument on the dissipation of the distance: for $0 \leq t \leq T$,

$$\|f_\varepsilon - \tilde{\rho}\mathcal{M}\|_{L^2_{x,v}(\mathcal{M}^{-1})} \leq e^{CT}\left(\|f_{\varepsilon,\mathrm{in}} - \rho_{\mathrm{in}}\mathcal{M}\|_{L^2_{x,v}(\mathcal{M}^{-1})} + C'\varepsilon(1 + T^{\frac{1}{2}})\right).$$

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- Taking $T \simeq -\log \varepsilon$ gives the result.

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Thank you!

On a class of nonlinear BGK-type kinetic equations with density dependent collision rates, J. Evans, M. D, H. Yoldaş, [arXiv:2503.14225]

Additional slides

Define the hydrodynamic quantities associated to f_ε by

$$\rho_\varepsilon := \int f_\epsilon \, \mathrm{d}v, \qquad j_\varepsilon := \int f_\varepsilon v \, \mathrm{d}v, \qquad E_\varepsilon := \int f_\varepsilon (v \otimes v - I) \, \mathrm{d}v,$$

then

$$\partial_t \rho_\varepsilon = -\frac{1}{\varepsilon} \nabla_x \cdot j_\varepsilon,$$
$$\partial_t j_\varepsilon = -\frac{1}{\varepsilon} \nabla_x \cdot E_\varepsilon - \frac{1}{\varepsilon} \nabla_x \rho_\varepsilon - \frac{1}{\varepsilon^2} \rho_\varepsilon^\alpha j_\varepsilon.$$

Lemma 1

For arbitrary $\beta \in \mathbb{R}$ we have

$$|j_\varepsilon| \leq \left(\int_{\mathbb{R}^d} \left(\frac{f_\varepsilon}{\mathcal{M}} - \beta \right)^2 \mathcal{M} \, \mathrm{d}v \right)^{\frac{1}{2}} \quad \text{and} \quad |E_\varepsilon| \leq \left(\int_{\mathbb{R}^d} \left(\frac{f_\varepsilon}{\mathcal{M}} - \beta \right)^2 \mathcal{M} \, \mathrm{d}v \right)^{\frac{1}{2}}.$$

Taking $\beta = \rho_\epsilon$ and integrating in space, time, we have

$$\|j_\varepsilon\|^2_{L^2([0,T]\times\mathbb{T}^d)}, \|E_\varepsilon\|^2_{L^2([0,T]\times\mathbb{T}^d)} \leq \int_0^T \|(I-\Pi)f_\varepsilon\|^2_{L^2_{x,v}(\mathcal{M}^{-1})}\,\mathrm{d}t.$$

By microscopic coercivity we know

$$\int_0^T \|(I-\Pi)f_\varepsilon\|^2_{L^2_{x,v}(\mathcal{M}^{-1})}\,\mathrm{d}t \lesssim \varepsilon^2 \|f_{\varepsilon,\mathrm{in}}\|^2_{L^2_{x,v}(\mathcal{M}^{-1})} \lesssim \varepsilon^2,$$

and so we have quantative rates of convergence to zero of $j_\varepsilon, E_\varepsilon$ in $L^2([0,T]\times T^d)$. We aim to make use of these in our short time asymptotics study.

Since $\partial_t \rho_\varepsilon = -\varepsilon^{-1} \nabla_x \cdot j_\varepsilon$ we expect that as $\varepsilon \to 0$, we should see

$$R_\varepsilon := \varepsilon^{-1} \rho_\varepsilon^\alpha j_\varepsilon + \nabla_x \rho_\varepsilon \to 0.$$

Rewriting R_ε we find that

$$-\varepsilon^{-1} \rho_\varepsilon^\alpha j_\varepsilon - \nabla_x \rho_\varepsilon = \varepsilon \partial_t j_\varepsilon + \nabla_x \cdot E_\varepsilon,$$

and so, in fact, we have a quantitative rate of convergence to zero of R_ε in $H^{-1}([0,T] \times \mathbb{T}^d)$. After lots of computation we find

$$\|f_\varepsilon - \tilde{\rho}\mathcal{M}\|^2_{L^2_{x,v}(\mathcal{M}^{-1})} \leq e^{Ct}\Big(\|f_{\varepsilon,\mathrm{in}} - \rho_{\mathrm{in}}\mathcal{M}\|^2_{L^2_{x,v}(\mathcal{M}^{-1})} + 2\int_0^t \int_{\mathbb{T}^d} R_\varepsilon \cdot \tilde{\rho}^{-\alpha} \nabla_x \tilde{\rho} \,\mathrm{d}x \,\mathrm{d}s\Big),$$

and so we may use the regularity of $\tilde{\rho}^{-\alpha} \nabla_x \tilde{\rho}$ to conclude.

Formal derivation

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Formally expand f_ε and ρ_{f_ε} as Hilbert series, $f_\varepsilon = f_0 + \varepsilon f_1 + O(\varepsilon^2)$, $\rho_{f_\varepsilon} = \rho_0 + \varepsilon\rho_1 + O(\varepsilon^2)$. In order to fully separate by powers of ε, we expand the power law as,

$$\rho_{f_\varepsilon}^\alpha = \rho_0^\alpha \left(1 + \varepsilon \frac{\rho_1}{\rho_0} + O(\varepsilon^2)\right)^\alpha = \rho_0^\alpha + \alpha\varepsilon\rho_0^{\alpha-1}\rho_1 + O(\varepsilon^2).$$

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Plugging everything in to (1) we find

$$\varepsilon v \cdot \nabla_x f_0 = \left(\rho_0^\alpha + \alpha\varepsilon\rho_0^{\alpha-1}\rho\right)\left(\rho_0\mathcal{M} - f_0 + \varepsilon(\rho_1\mathcal{M} - f_1)\right) + O(\varepsilon^2).$$

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Formally we find at lowest order that

ε^0: $$\rho_0^\alpha(\rho_0\mathcal{M} - f_0) = 0 \implies f_0 = \rho_0\mathcal{M}.$$

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Next order of ε gives

$$\varepsilon^1: \qquad v\cdot\nabla_x f_0 = \rho_0^\alpha\left(\mathcal{M}\rho_1 - f_1\right) \implies f_1 = \mathcal{M}\rho_1 - v\mathcal{M}\cdot\rho_0^{-\alpha}\nabla_x\rho_0.$$

To close the system we look at the conservation law for ρ_{f_ε},

$$\partial_t \rho_{f_\varepsilon} + \frac{1}{\varepsilon} \nabla_x \cdot \int_{\mathbb{R}^d} v f_\varepsilon \, \mathrm{d}v = 0.$$

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Plugging in previous calculations, we find that at order ε^0,

ε^0: $$\partial_t \rho_0 = -\nabla_x \cdot \int_{\mathbb{R}^d} v f_1 \, \mathrm{d}v = \nabla_x \cdot \left(\rho_0^{-\alpha} \nabla_x \rho_0 \right),$$

thus we formally conclude that, as $\varepsilon \to 0$, we have $f_\varepsilon \to \rho_0 \mathcal{M}$ where,

$$\partial_t \rho_0 = \nabla_x \cdot \left(\rho_0^{-\alpha} \nabla_x \rho_0 \right).$$

Lemma 2

Define $\bar{u} = \int u \, \mathrm{d}x$. Then the following estimates hold for $(I - \Delta_x)^{-1}$,

$$\begin{aligned}
\|(I - \Delta_x)^{-1} \partial^2_{x_i, x_j} u\|^2_{L^2_x} &\leq \|u\|^2_{L^2_x}, \\
\|(I - \Delta_x)^{-1} \partial_{x_i} u\|^2_{L^2_x} &\leq \|u - \bar{u}\|^2_{L^2_x} \leq \|u\|^2_2, \\
\|(I - \Delta_x)^{-1/2} \nabla_x u\|^2_{L^2_x} &\geq \|u - \bar{u}\|^2_{L^2_x}.
\end{aligned}$$

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Let $\varepsilon > 0, A > 1, \alpha \in \mathbb{R}$, and let f_ε be the solution of (1) with $\|f_\varepsilon\|_{L^1_{x,v}} = 1$. Suppose that $\rho_{f_\varepsilon} = \int_{\mathbb{R}^d} f_\varepsilon \, \mathrm{d}v$ satisfies $A^{-1} \leq \rho_{f_\varepsilon} \leq A$. Then there exists $\varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$ then

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Theorem 2 [*Long-time behaviour*]

Let $\varepsilon > 0, A > 1, \alpha \in \mathbb{R}$, and let f_ε be the solution of (1) with $\|f_\varepsilon\|_{L^1_{x,v}} = 1$. Suppose that $\rho_{f_\varepsilon} = \int_{\mathbb{R}^d} f_\varepsilon \, \mathrm{d}v$ satisfies $A^{-1} \leq \rho_{f_\varepsilon} \leq A$. Then there exists $\varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$ then

$$\|f_\varepsilon - \mathcal{M}\|^2_{L^2_{x,v}(\mathcal{M}^{-1})} \leq 3e^{-\gamma t}\|f_{\varepsilon,\mathrm{in}} - \mathcal{M}\|^2_{L^2_{x,v}(\mathcal{M}^{-1})}, \tag{5}$$

where $\gamma > 0$ is a constant depending only on α, A.

Sketch of proof

- We adapt linear L^2 hypocoercivity methods[3] to our nonlinear equation.
- First show microscopic coercivity,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f_\varepsilon - \mathcal{M}\|^2_{L^2_{x,v}(\mathcal{M}^{-1})} \leq -\frac{2}{\varepsilon^2 A^\alpha}\|(I - \Pi)f_\varepsilon\|^2_{L^2_{x,v}(\mathcal{M}^{-1})}, \quad \Pi f := \rho_f \mathcal{M},$$

[3] J. Dolbeault, C. Mouhot, and C. Schmeiser (June 2015). "Hypocoercivity for linear kinetic equations conserving mass". In: *Trans. Amer. Math. Soc.* 367.6, pp. 3807–3828. ISSN: 0002-9947. DOI: 10.1090/S0002-9947-2015-06012-7

- Now add a macroscopic perturbation term,

$$I_\varepsilon := \int_{\mathbb{T}^d} j_\varepsilon \cdot (I - \Delta_x)^{-1} \nabla_x \rho_\varepsilon \, \mathrm{d}x,$$

and show there exist $\gamma, C > 0$, depending only on α, A, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|f_\varepsilon - \mathcal{M}\|^2_{L^2_{x,v}(\mathcal{M}^{-1})} + \varepsilon C I_\varepsilon\right) \leq -\gamma\left(\|f_\varepsilon - \mathcal{M}\|^2_{L^2_{x,v}(\mathcal{M}^{-1})} + \varepsilon C I_\varepsilon\right).$$

- Conclude with Grönwall and

$$\varepsilon C |I_\varepsilon| \leq \frac{1}{2} \|f_\varepsilon - \mathcal{M}\|^2_{L^2_{x,v}(\mathcal{M}^{-1})}.$$

Theorem 3 [*Diffusive asymptotics*]

Let $A > 1$ and $\varepsilon \in (0, \varepsilon_0)$ where ε_0 given by Theorem 2. Consider a sequence of functions $\{f_{\varepsilon,\mathrm{in}}\} \subset L^1(\mathbb{T}^d \times \mathbb{R}^d)$ satisfying $\|f_{\varepsilon,\mathrm{in}}\|_{L^1} = 1$ and $A^{-1}\mathcal{M} \leq f_{\varepsilon,\mathrm{in}} \leq A\mathcal{M}$, and let f_ε be the corresponding solution of (1). Let $\rho_{\mathrm{in}} \in C^2(\mathbb{T}^d)$ satisfy $\|\rho_{\mathrm{in}}\|_{L^1} = 1$, $A^{-1} \leq \rho_{\mathrm{in}} \leq A$, and let $\tilde{\rho}$ be the corresponding solution of (2). Suppose the initial data is well-prepared,

$$\|f_{\varepsilon,\mathrm{in}} - \rho_{\mathrm{in}}\mathcal{M}\|_{L^2_{x,v}(\mathcal{M}^{-1})} \leq \varepsilon,$$

then there exist constants $C, \kappa > 0$ depending only on $\alpha, A, \|\rho_{\mathrm{in}}\|_{C^2}, d$ such that

$$\|f_\varepsilon - \tilde{\rho}\mathcal{M}\|_{L^\infty(\mathbb{R}_+; L^2_{x,v}(\mathcal{M}^{-1}))} \leq C\varepsilon^\kappa. \tag{6}$$

Sketch of proof

- Following Anceschi, Zhu[4] we study the asymptotics on short times and long times separately.

[4] F. Anceschi and Y. Zhu (2021). "On a spatially inhomogeneous nonlinear Fokker–Planck equation : Cauchy problem and diffusion asymptotics". In: *Analysis & PDE*

Sketch of proof

- Following Anceschi, Zhu[4] we study the asymptotics on short times and long times separately.
- On long times we combine exponential relaxation to equilibrium of both solutions: for $t \geq T$,

$$\|f_\varepsilon - \tilde{\rho}\mathcal{M}\|_{L^2_{x,v}(\mathcal{M}^{-1})} \leq \|f_\varepsilon - \mathcal{M}\|_{L^2_{x,v}(\mathcal{M}^{-1})} + \|\tilde{\rho} - 1\|_{L^2_x} \lesssim e^{-\gamma T}.$$

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- On short times we write a Grönwall argument on the dissipation of the distance: for $0 \leq t \leq T$,

$$\|f_\varepsilon - \tilde{\rho}\mathcal{M}\|_{L^2_{x,v}(\mathcal{M}^{-1})} \leq e^{CT}\left(\|f_{\varepsilon,\mathrm{in}} - \rho_{\mathrm{in}}\mathcal{M}\|_{L^2_{x,v}(\mathcal{M}^{-1})} + C'\varepsilon(1 + T^{\frac{1}{2}})\right).$$

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- Taking $T \simeq -\log\varepsilon$ gives the result.

[4] F. Anceschi and Y. Zhu (2021). "On a spatially inhomogeneous nonlinear Fokker–Planck equation : Cauchy problem and diffusion asymptotics". In: *Analysis & PDE*