Maximum Principles
and
Geometric Applications

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To my parents,
to my brothers,
to Ale.
To my family and friends.

If you can dream it,
you can do it.

Walt Disney

Preface

These notes are the main body of the course *Maximum Principles and Geometric Applications* to be hold in Brazilia at the XVIII Escola de Geometria Diferencial.

We would like to introduce to the students a fundamental tool in Partial Differential Equations, the Maximum Principle for elliptic equations, and its important applications in Geometry, probably the Alexandrov Reflection Method is the major one. We use the Alexandrov Reflection Method for classifying properly embedded constant mean curvature surfaces of finite topology in the Euclidean Space.

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Chapter 1

Introduction

In 1927, H. Hopf [11] extended the Maximum Principle for harmonic functions, that is, a harmonic function can not have an interior maximum unless it is constant, to more general elliptic partial differential equations. The Maximum Principle is based on the following observation:

> *Let $\Omega \subset \mathbb{R}^2$ be a domain, given a smooth function u defined on Ω, if $u \in C^2(\Omega)$ and it has a maximum at a point $x_0 \in \Omega$, then $\nabla u(x_0) = 0$ and $\nabla^2 u(x_0) \leq 0$, where ∇u and $\nabla^2 u$ are the gradient and the Hessian of u at the point $x_0 \in \Omega \subset \mathbb{R}^2$. In particular, this implies that a harmonic function can not have an interior maximum unless it is constant.*

In fact, he was able to prove:

> **Interior Maximum Principle [12]:** *Let $\Omega \subset \mathbb{R}^2$ be a domain and*
>
> $$Lu \equiv \sum_{i,j=1}^{2} a_{ij}(x)\partial_{ij}^2 u + \sum_{i=1}^{2} b_i(x)\partial_i u + c(x)u$$
>
> *be an uniformly elliptic differential operator on Ω. Suppose that $Lu \geq 0$ for a function $u \in C^2(\Omega)$. Then,*
>
> - *if $c \equiv 0$ and u has a maximum in Ω, then u is constant;*
> - *if $c \leq 0$ and u has a non-negative maximum in Ω, then u is constant.*

And

> **Boundary Maximum Principle [12]:** *Let $\Omega \subset \mathbb{R}^2$ be a domain such that $\partial\Omega$ is smooth, and L be an uniformly elliptic differential operator on Ω. Let $x_0 \in \partial\Omega$ so that*

- *u is C^1 at x_0;*
- *$u(x_0) \geq u(x)$, for all $x \in \Omega$;*
- *$\frac{\partial u}{\partial \eta}(x_0) = 0$, where η is the inward normal to $\partial\Omega$.*

Then,

- *if $c \equiv 0$, u is constant;*
- *if $c \leq 0$ and $u(x_0) \geq 0$, u is constant.*

The Maximum Principle had an amazing impact in the theory of partial differential equations, becoming an essential tool in the area until today. Not only it was important in PDEs but in Geometry too. To see so, we focus on its geometric interpretation.

Given a surface Σ in the Euclidean Space $\mathbb{R}^3$, locally we can represent Σ as the graph of a function $u : \Omega \to \mathbb{R}$ over a domain on its tangent space, one can consider $\Omega \subset \mathbb{R}^2$. Moreover, if Σ satisfies certain condition on its curvatures, say constant mean curvature for example, u satisfies an elliptic partial differential equation.

This is a fundamental observation that let to A.D. Alexandrov to prove:

Alexandrov's Theorem [2]: *A compact embedded constant mean curvature surface Σ in $\mathbb{R}^3$ must be a round sphere.*

Alexandrov's idea was to compare the original surface with its reflection with respect to planes, today, such a method is known as the *Alexandrov Reflection Method*. Recall that reflections with respect to totally geodesic planes are isometries in $\mathbb{R}^3$ and therefore, the reflection preserves the constancy of the mean curvature. Let us explain the idea in more detail.

Take a plane P not intersecting Σ and approach P to Σ until its first contact point. At this point, Σ is completely contained in one of the closed halfspace determined by P. If we keep moving P in this direction, for a small displacement, a small region of Σ belongs now to the halfspace that was empty. If we reflect this small part of Σ with respect to this plane P, then the reflected part is contained in the bounded domain determined by Σ. So, if we continue to move P and reflecting Σ, there must be a traslation of P so that the reflected part of Σ left the bounded domain determined by Σ. In other words, Σ and its reflection by this plane have a tangent point, either at the interior or at the boundary. Let us explain what this means.

First, the geometrical meaning of an interior tangent point. Let $\Sigma_i \subset \mathbb{R}^3$ be surfaces, $i = 1, 2$, and $p \in \operatorname{int}(\Sigma_1) \cap \operatorname{int}(\Sigma_2)$ such that the tangent plane and the normal unit vector of Σ_1 and Σ_2 at p coincide. In these conditions, we say that p is an *interior tangent point of Σ_1 and Σ_2* (see Figure 1.1).

Second, the geometrical meaning of a boundary tangent point. Let $\Sigma_i \subset \mathbb{R}^3$ be surfaces with smooth $\partial\Sigma_i \neq \emptyset$, $i = 1, 2$. Let $p \in \partial\Sigma_1 \cap \partial\Sigma_2$ be a point such that the tangent planes and the unit normal vector of Σ_1 and Σ_2, and the interior conormal vectors of $\partial\Sigma_1$ and $\partial\Sigma_2$, coincide at p. In this conditions, we say that p is a *boundary tangent point of Σ_1 and Σ_2*.

Observe that if p is either an interior or a boundary tangent point of $\Sigma_1 = graph(u_1)$ and $\Sigma_2 = graph(u_2)$, then u_1 and u_2 are defined over the same tangent plane. In this case, we say that Σ_1 is *above* (resp. *below*) of Σ_2, if $u_1 \geq u_2$ (resp. $u_1 \leq u_2$).

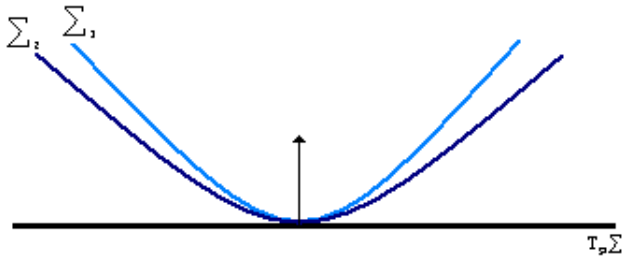


Figure 1.1: Interior tangent point.

Therefore, since Σ and its reflection by this plane have a tangent point, either at the interior or at the boundary, we can apply the Hopf Maximum Principle, either for the interior point or the boundary point, to conclude that both surfaces are the same, that is, we have found a plane of symmetry for Σ. But we can do this in any direction, so Σ must be a round sphere.

These two magnificent results show the close interaction between partial differential equations and geometry. In this book we will exploit the geometric applications of the Maximum Principle for elliptic partial differential equations.

In Chapter 2, we give a detailed proof of the Hopf Maximum Principle for linear elliptic partial differential equations. We continue with the more general quasilinear elliptic equations, that will fit most of the geometric problems we are interested on. In particular, a quasilinear elliptic equation appears for constant mean curvature surfaces. Hence, we finish this chapter by proving the Tangency Principle for quasilinear equations, that is,

> **Tangency Principle:** *Let $u^k : U \to \mathbb{R}$, $k = 1, 2$, be two C^2 function defined on an open domain U of either $\mathbb{R}^2$ or the halfspace $\left\{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0\right\}$ such that $(0, 0) \in U$.*
>
> *Assume u^k, $k = 1, 2$, is a solution to the same quasilinear elliptic equation $Q(u^k) = 0$. Then,*
>
> - *if $(0, 0)$ is an interior point at U, $u^1(0, 0) = u^2(0, 0)$ and $u^1 \leq u^2$ in U, then $u^1 \equiv u^2$ in U,*

- *if* $(0,0)$ *is a boundary point at* U, $u^1(0,0) = u^2(0,0)$, $\partial_2 u^1(0,0) = \partial_2 u^2(0,0)$ *and* $u^1 \leq u^2$ *in* U, *then* $u^1 \equiv u^2$ *in* U.

The proof of the above Tangency Principle is based on the fact that the difference of two solutions of the same quasilinear elliptic equation satisfies a linear elliptic equation and, therefore, applying the Hopf Maximum Principle we obtain the result.

In Chapter 3 we focus on the study of properly embedded constant mean curvature surfaces, $H \neq 0$, of finite topology. The main tool will be the geometric version of the Tangency Principle.

> **Geometric Maximum Principle:** *Let* Σ_i, $i = 1, 2$, *be surfaces in* $\mathbb{R}^3$ *with the same constant mean curvature* H. *Assume that* p *is an interior or boundary tangent point of* Σ_1 *and* Σ_2 *and, in a neighborhood of* p, $\Sigma_1 \geq \Sigma_2$. *Then* $\Sigma_1 = \Sigma_2$.

With the Geometric Maximum Principle in hands, we will classify properly embedded constant mean curvature surfaces of finite topology in the Euclidean Space. In fact, we will study more general families of surfaces that satisfies the Geometric Maximum Principle but, to keep it simple in the Introduction, we focus only on constant mean curvature surfaces in the Euclidean Space.

We will use the Geometric Maximum Principle to introduce the Alexandrov Reflection Method and we prove the Alexandrov's Theorem. We continue by giving geometric height estimates for graphs of constant mean curvature surfaces with boundary on a plane. This will lead us to a geometric height estimates for embedded compact constant mean curvature surfaces with boundary in a plane. At this point, we have the necessary tools for studying properly embedded constant mean curvature surfaces [13, 14]. So, we will prove:

> **Classification Properly Embedded CMC-surfaces of Finite Topology:** *Let* $\Sigma \in \mathbb{R}^3$ *be a properly embedded surface with finite topology in* $\mathbb{R}^3$ *of constant mean curvature* $H \neq 0$.
>
> *Then every end of* Σ *is cylindrically bounded. Moreover, if* $a_1, \ldots, a_k$ *are the* k *axial vectors corresponding to the ends, then these vectors cannot be contained in an open hemisphere of* $\mathbb{S}^2$. *In particular,*
>
> - $k = 1$ *is impossible.*
> - *If* $k = 2$, *then* Σ *is contained in a cylinder and is a rotational surface with respect to a line parallel to the axis of the cylinder (see Figure 3.8).*
> - *If* $k = 3$, *then* Σ *is contained in a slab.*

Chapter 2

Maximum Principle for Elliptic Equations

Elliptic Maximum Principles are in the core of geometric problems. The Maximum Principle is based in the following observation. Let $\Omega \subset \mathbb{R}^2$ be a domain, given a smooth function u defined on Ω, if $u \in C^2(\Omega)$ and it has a maximum at a point $x_0 \in \Omega$, then $\nabla u(x_0) = 0$ and $\nabla^2 u(x_0) \leq 0$, where ∇u and $\nabla^2 u$ are the gradient and the Hessian of u at the point $x_0 \in \Omega \subset \mathbb{R}^2$. In particular, this implies that a harmonic function can not have an interior maximum unless it is constant.

We will be interested on more general differential equations than the harmonic equation, equations related to geometric aspects of immersed surfaces in $\mathbb{R}^3$. We will give a detailed proof of the Hopf Maximum Principle for linear elliptic second order equations. Later, we will reduce the quasilinear elliptic second order equation case, the one that has geometric applications in our case, to the linear case and therefore we will prove the Tangency Principle.

We remind the basic facts on PDEs. We follow [10, Chapter 10].

2.1 The Maximum Principle for Linear Elliptic Equations

We will recapture here the Classical Maximum Principle of H. Hopf [12] for second order linear elliptic equations (see [9, 10, 18]). We consider **second order linear operators** of the form

$$\sum_{i,j=1}^{n} a_{ij}\, \partial_{ij}^2 u + \sum_{i=1}^{n} b_i\, \partial_i u + c\, u = 0, \tag{2.1}$$

where $x = (x_1, \ldots, x_n) \in \Omega \subset \mathbb{R}^n$ and a_{ij}, b_i and c are continuous functions defined on Ω, i.e., $a_{ij}, b_i, c \in C^0(\Omega)$. Here, $\partial_i u$ denotes the partial derivative of u in the x_i-direction, that is, $\partial_i u = \frac{\partial u}{\partial x_i}$, and $\partial_{ij}^2 u = \frac{\partial^2 u}{\partial x_i \partial x_j}$. Moreover, we ask for the matrix $A = (a_{ij})_{i,j=1,\ldots,n} \in \mathcal{M}(n \times n)$ to be symmetric. Here $\mathcal{M}(n \times n)$ represents the space of matrizes with n rows and n columns.

Remark 1:
For most of the results, it is enough to ask that b_i and c are locally bounded.

Moreover, unless otherwise stated, it will be assumed that $u \in C^\infty(\Omega)$. Nevertheless, the differentiability hypothesis assumed here are much more restrictive than it really requires, $u \in C^2$ is enough (see [10, Chapter 3]).

We can represent (2.1) as

$$Lu = 0,$$

where $L : C^2(\Omega) \to C^0(\Omega)$ is the differential operator defined by

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}\, \partial_{ij}^2 u + \sum_{i=1}^{n} b_i\, \partial_i u + c\, u. \tag{2.2}$$

We adopt the following definitions:

Definition 2.1:
Let L be a differential operator as in (2.2). Then, we say L is

- ***Elliptic** if all the eigenvalues of the matrix A are positive.*
- ***Uniformly Elliptic** if the eigenvalues of the matrix A are bounded below and above by a positive constant.*
- ***Hyperbolic** if all the eigenvalues of A are non-vanishing, but there exist positive and negative eigenvalues.*
- ***Parabolic** if some eigenvalues vanishes.*

Let us denote by

$$m(x) = \min\{a_{ij}(x) \,:\, i,j = 1,\dots,n\}$$

and

$$M(x) = \max\{a_{ij}(x) \,:\, i,j = 1,\dots,n\},$$

for $x \in \Omega$. Then, ellipticity means that

$$0 < m(x)|\xi|^2 \leq \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \leq M(x)|\xi|^2 \ \forall x \in \Omega, \tag{2.3}$$

where $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $|\cdot|$ denotes the usual Euclidean norm. Moreover, if M/m is bounded then L is uniformly elliptic.

2.1.1 The Hopf Maximum Principle

We must bear in mind that the simplest case of a second order linear operator is the Laplacian Δ. If $u \in C^2(\Omega)$ has a maximum at $x_0 \in \Omega$, it is a simple observation that $\Delta u(x_0) \leq 0$. For a general linear second order elliptic operator we can show:

Proposition 2.2:
Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying $Lu > 0$ on Ω, L is an elliptic second order linear operator given by (2.2) with $c \leq 0$. Then,

- *if $c \equiv 0$, u does not have a maximum in Ω;*
- *if $c \leq 0$, u does not have a nonnegative local maximum in Ω.*

Proof. Let $x_0 \in \Omega$ be an interior local maximum, then the Hessian matrix of u at x_0, $\nabla^2 u(x_0)$, is negative semidefinite. We will prove the case $c \leq 0$ since it is more general. We will assume that $u(x_0) \geq 0$ and we will get a contradiction.

Since L is elliptic, $A := A(x_0) = (a_{ij}(x_0))_{i,j=1,\dots,n} \in \mathcal{M}(n \times n)$ is symmetric and positive definite and so, there exists an orthogonal matrix that diagonalizes A, that is,

$$PAP^{-1} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix},$$

where $\lambda_i > 0$, $i = 1, \dots, n$. A simple computation shows that

$$\mathrm{Tr}(A\nabla^2 u(x_0)) = \sum_{i,j=n}^{2} a_{ij}(x_0)\partial^2_{ij} u(x_0) = Lu(x_0) - c(x_0)u(x_0) > 0$$

since $\partial_i u(x_0) = 0$ and the hypothesis on $Lu > 0$ and $c \leq 0$. Here Tr stands for the Trace Operator.

Therefore, if we show that $\mathrm{Tr}(A\nabla^2 u(x_0)) \leq 0$, we get the desired contradiction and prove the result. Since the trace is invariant for similar matrices, we have

$$\begin{aligned} \mathrm{Tr}(A\nabla^2 u(x_0)) &= \mathrm{Tr}(PA\nabla^2 u(x_0)P^{-1}) = \mathrm{Tr}(PAP^{-1}P\nabla^2 u(x_0)P^{-1}) \\ &= \mathrm{Tr}\left(\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_2 \end{pmatrix} C\right) = \sum_{i=1}^{n} \lambda_i c_{ii}, \end{aligned}$$

where $C := P\nabla^2 u(x_0)P^{-1}$. Since $\nabla^2 u(x_0)$ is negative semidefinite so is C and therefore $c_{ii} \leq 0$, $i = 1, \dots, n$. Thus,

$$\mathrm{Tr}(A\nabla^2 u(x_0)) = \sum_{i=1}^{n} \lambda_i c_{ii} \leq 0$$

and the result is proved. □

We can relax the hypothesis $Lu > 0$ by impossing more conditions on the ellipticity of L. Precisely we will ask

Definition 2.3:
Let $\Omega \subset \mathbb{R}^2$ be a domain. A second order linear operator L given by (2.2) is said to be **locally uniformly elliptic** *if for any $x_0 \in \Omega$ there exist a neighborhood $x_0 \in U \subset \Omega$ of x_0 and positive constants $\lambda(x_0)$ and $\Lambda(x_0)$ such that*

$$\lambda(x_0)|\xi|^2 \leq \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \leq \Lambda(x_0)|\xi|^2 \ \forall x \in U,\ \xi \in \mathbb{R}^2.$$

Now we can announce

Theorem 2.4 (Interior Maximum Principle [12])**:**
Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying $Lu \geq 0$ on Ω, L is a locally uniformly elliptic second order linear operator given by (2.2).

- *If $c \equiv 0$ and u has a local maximum in Ω, then u is locally constant.*
- *If $c \leq 0$ and u has a nonnegative local maximum in Ω, then u is locally constant.*

Proof. Let $x_0 \in \Omega$ be a local interior maximum. Since L is locally uniformly elliptic, we can find $r > 0$ such that

$$u(x) \leq u(x_0)$$

and

$$\lambda(x_0)|\xi|^2 \leq \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \leq \Lambda(x_0)|\xi|^2$$

for all $x \in B(x_0, r) \subset \Omega$, where $B(x_0, r)$ denotes the Euclidean ball centered at x_0 of radius r.

The proof will be by contradiction. Assume that u is nonconstant in any neighborhood of x_0. Then, there exist

- $y \in B(x_0, r/2)$ such that $u(y) < u(x_0)$, and
- $\delta_0 > 0$ such that $\overline{B(y, \delta_0)} \subset B(x_0, r)$ and $x_0 \in \partial B(y, \delta_0)$.

Set $U(x_0) = \{x \in \Omega : u(x) = u(x_0)\}$ and define

$$\delta = \inf\left\{\rho > 0 \ : \ \overline{B(y, \rho)} \subset B(x_0, r) \text{ and } \partial B(y, \rho) \cap U(x_0) \neq \emptyset\right\}.$$

One can easily check that $0 < \delta \leq \delta_0$. This follows since there exists $0 < \tilde{\delta} < \delta_0$ such that $u(x) < u(x_0)$ for all $x \in \overline{B(y, \tilde{\delta})}$ by continuity. So, by construction, $u(x) < u(x_0)$ for all $x \in \overline{B(y, \delta)}$ and there exists $x' \in \partial B(y, \delta)$ such that $u(x') = u(x_0)$.

Take $0 < \delta' < \delta$ so that $\overline{B(x', \delta')} \subset B(x_0, r)$ and a point $p \in [y, x'] = \{sy + (1-s)x' : s \in [0,1]\}$ so that $|p - x'| > \delta'$. In particular, $u(x) \leq u(x_0)$ for all $x \in B(x', \delta')$.

Let K be a constant to be determined and define

$$v(x) = e^{-K|x-p|^2} - e^{-K|x'-p|^2},\ x \in \overline{B(x', \delta')}, \tag{2.4}$$

then, a straightforward computation shows that

$$\begin{aligned}
Lv(x) = e^{-K|x-p|^2} &\left(4K^2 \sum_{i,j=1}^{2} a_{ij}(x)\,(x_i - p_i)(x_j - p_j) - 2K \sum_{i=1}^{2} a_{ij}(x) \right) \\
&- 2Ke^{-K|x-p|^2} \sum_{i=1}^{2} b_i(x)\,(x_i - p_i) + c(x)v(x) = \\
= 4K^2 e^{-K|x-p|^2} &\left(\sum_{i,j=1}^{2} a_{ij}(x)\,(x_i - p_i)(x_j - p_j) \right) \\
&- 2Ke^{-K|x-p|^2} \left(\sum_{i=1}^{2} a_{ij}(x) + \sum_{i=1}^{2} b_i(x)\,(x_i - p_i) \right) \\
&+ e^{-K|x-p|^2} \left(c(x) - c(x)e^{K(|x-p|^2 - |x'-p|^2)} \right) = \\
= e^{-K|x-p|^2} &\left(4K^2 M(x) - 2KN(x) + P(x) \right),
\end{aligned}$$

where

$$\begin{aligned}
M(x) &:= \sum_{i,j=1}^{2} a_{ij}(x)\,(x_i - p_i)(x_j - p_j), \\
N(x) &:= \sum_{i=1}^{2} a_{ij}(x) + \sum_{i=1}^{2} b_i(x)\,(x_i - p_i), \\
P(x) &:= c(x) - c(x)e^{K(|x-p|^2 - |x'-p|^2)},
\end{aligned}$$

Since $\overline{B(x', \delta')}$ is compact, we can find constants $M, N, P \in \mathbb{R}$, independent of K, such that

$$Lv(x) \geq e^{-K|x-p|^2} \left(4K^2 M - 2KN + P \right) \text{ for all } x \in \overline{B(x', \delta')},$$

where $M > 0$ since L is locally uniformly elliptic.

Note that $P(x)$ is bounded below by a constant independent of K, given by

$$c_1 := \min\left\{c(x) : x \in \overline{B(x', \delta')}\right\},$$

in the case $c \equiv 0$, we have $P \equiv 0$. Moreover, $N(x)$ is bounded above since b_i and a_{ij} are continuous functions.

So, since $M > 0$, one can find $K > 0$ such that

$$Lv(x) > 0 \text{ for all } x \in \overline{B(x', \delta')}.$$

Now, for all $t > 0$ we have

$$L(u + tv) > 0 \text{ in } B(x', \delta').$$

Let $\partial B(x', \delta') = \partial_1 \cup \partial_2$, where $\partial_1 = \partial B(x', \delta') \cap \overline{B(p, |x' - p|)}$ and $\partial_2 = \partial B(x', \delta') \setminus \partial_1$. Note that

- if $x \in \partial_2$, then $|x - p| > |x' - p|$ and so $v(x) < 0$,
- if $x \in \partial_1$, then $u(x) < u(x_0)$,

and since ∂_1 is compact, we can find $t > 0$ such that

$$u(x) + tv(x) \leq u(x_0) \text{ for all } x \in \partial B(x', \delta').$$

Therefore, there exists $y \in B(x', \delta')$ so that

$$u(y) + tv(y) \geq u(x) + tv(x) \text{ for all } x \in \overline{B(x', \delta')}.$$

Moreover, since $u(x') + tv(x') = u(x') = u(x_0)$, we obtain

$$u(y) + tv(y) \geq u(x_0) \geq 0.$$

Thus, applying Proposition 2.2 to $u + tv$ in $B(x', \delta')$ we get a contradiction. Hence, there exists $r > 0$ such that $u(x) = u(x_0)$ for all $x \in B(x_0, r)$. □

Next, we will study the case when the maximum is attained at the boundary.

Theorem 2.5 (Boundary Maximum Principle [12])**:**
Let $\Omega \subset \mathbb{R}^2$ be a domain such that $\partial\Omega$ is smooth and L be a locally uniformly elliptic differential operator on Ω. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying $Lu \geq 0$ on Ω. Let $x_0 \in \partial\Omega$ so that

- *u is C^1 at x_0;*
- *$u(x_0) \geq u(x)$ for all $x \in \Omega \cap B(x_0, \epsilon)$, for some $\epsilon > 0$;*

- $\frac{\partial u}{\partial \eta}(x_0) \geq 0$, *where* η *is the inward normal to* $\partial\Omega$.

Then,

- *if* $c \equiv 0$, u *locally is constant;*
- *if* $c \leq 0$ *and* $u(x_0) \geq 0$, u *is locally constant.*

Proof. Assume that u is non constant in a neighborhood of x_0. Since $\partial\Omega$ is smooth, there exists $\rho > 0$ and $x' \in \Omega$ so that $B(x', \rho) \subset \Omega$ and $x_0 \in \partial B(x', \rho)$.

Let $\rho' < \min\{\rho, \epsilon\}$ be such that $u(x) \leq u(x_0)$ for all $x \in B(x_0, \rho') \cap (\Omega \cup \{x_0\})$. Consider the compact set

$$K := \left\{x \in \Omega : \ |x - x_0| \leq \rho', \, |x - x'| \leq \rho\right\},$$

and define

$$v(x) = e^{-\delta|x-x'|^2} - e^{-\delta|x-x_0|^2} \text{ for all } x \in K,$$

here δ is a constant to be determined.

The same way we did in Theorem 2.4, one cand find $\delta > 0$ so that

$$Lv(x) > 0 \text{ for all } x \in K.$$

Take $\tilde{\rho} < \rho'$ so that $B(x_0, \tilde{\rho}) \cap B(x', \tilde{\rho}) = \emptyset$ and consider

$$\tilde{K} := \left\{x \in \Omega : \ |x - x_0| \leq \tilde{\rho}, \, |x - x'| \leq \rho\right\},$$

write $\partial\tilde{K} = \partial_1 \cup \partial_2$, where $\partial_1 = \overline{\partial\tilde{K} \cap B(x', \rho)}$ and $\partial_2 = \partial\tilde{K} \setminus \partial_1$. Therefore,

- If $x \in \partial_2$, then $v(x) = 0$.
- If $x \in \partial_1$, then $u(x) < u(x_0)$. Otherwise, there would exist $y \in \partial_1$ so that $u(y) = u(x_0)$, that is, y would be a nonnegative local maximum. Therefore, from Theorem 2.4, u would be constant in a neighborhood of y and so, constant in $B(x_0, \rho') \cap \Omega$, which is a contradiction.

Then, we can take $t > 0$ such that

$$u(x) + tv(x) \leq u(x_0) \text{ for all } x \in \partial_1.$$

Thus,

$$u(x) + tv(x) \leq u(x_0) \text{ for all } x \in \partial\tilde{K}$$

and

$$L\left(u - u(x_0) + tv\right) = Lu + tLv - c\,u(x_0) > 0 \text{ in } \tilde{K}.$$

Since $u(x) - u(x_0) + tv(x) \leq 0$ for all $x \in \partial\tilde{K}$, from Proposition 2.2, we have

$$u - u(x_0) + tv \leq 0 \text{ in } \tilde{K}.$$

Therefore, since $v(x_0) = 0$, we get

$$-t\frac{\partial v}{\partial \eta}(x_0) \geq \frac{\partial u}{\partial \eta}(x_0) \geq 0.$$

Nevertheless,

$$-t\frac{\partial v}{\partial \eta}(x_0) = -2t\delta\rho e^{-\delta\rho^2} < 0,$$

which is a contradiction. □

Actually, it is not necessary to restrict ourself to second order linear differential operators. In fact, it is well known that there are higher order differential operators satisfying the conclusions of the Interior and Boundary Maximum Principle (see [9] or [10]) as we will see in the following. To do so, we will need the following

Corollary 2.6:
Let $\Omega \subset \mathbb{R}^2$ be a domain such that $\partial\Omega$ is smooth, and let

$$Lu = \sum_{i,j=1}^{2} a_{ij}(x)\partial_{ij}^2 u + \sum_{i=1}^{2} b_i(x)\partial_i u + c(x)u,$$

be a locally uniformly elliptic differential operator on Ω, here c is assumed to be locally bounded.
Assume $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ so that $Lu \geq 0$ and $u \leq 0$ in Ω. Let $x_0 \in \overline{\Omega}$, then:

- *If $x_0 \in \Omega$ and $u(x_0) = 0$, then $u \equiv 0$ on $\overline{\Omega}$.*
- *If $x_0 \in \partial\Omega$, $u(x_0) = 0$, u is C^1 at x_0 and $\frac{\partial u}{\partial \eta}(x_0) \geq 0$, where η is the inward normal to $\partial\Omega$, then $u \equiv 0$ on $\overline{\Omega}$.*

Proof. Set

$$\Lambda = \{x \in \Omega \,:\, u(x) = 0\},$$

which is clearly a closed set since u is a continuous function.

Consider

$$L_0 u = \sum_{i,j=1}^{2} a_{ij}(x)\partial_{ij}^2 u + \sum_{i=1}^{2} b_i(x)\partial_i u,$$

and set $q(x) := \min\{c(x), 0\} \leq 0$ for all $x \in \Omega$. Therefore, since $q - c \leq 0$ in Ω, we obtain

$$L_0 u + qu = Lu + (q - c)u \geq 0 \text{ in } \Omega.$$

On the one hand, if $x_0 \in \Lambda$ applying Theorem 2.4 to $L_0 u + qu$ we obtain that there exists $\rho > 0$ such that

$$u(x) = 0 \text{ for all } x \in B(x_0, \rho) \subset \Omega,$$

that is, Λ is an open set. Therefore, since Ω is connected, we obtain that $\Lambda = \Omega$ and u is constant in Ω.

On the other hand, if $x_0 \in \partial\Omega$, $u(x_0) = 0$, u is C^1 at x_0 and $\frac{\partial u}{\partial \eta}(x_0) \geq 0$, where η is the inward normal to $\partial\Omega$, applying Theorem 2.5 to $L_0 u + qu$ we obtain $\Lambda \neq \emptyset$, and hence $\Lambda = \Omega$ as we did above. □

Remark 2:
We can relax the hypothesis on the boundary $\partial\Omega$ by asking only the condition of the interior tangent sphere. Instead to ask $\partial\Omega$ be smooth, one can ask that there exists a point $\tilde{x} \in \Omega$ and $\tilde{\rho} > 0$ such that

$$B(\tilde{x}, \tilde{\rho}) \subset \Omega \text{ and } x_0 \in \partial B(\tilde{x}, \tilde{\rho}).$$

2.2 The Maximum Principle for Quasilinear Elliptic Equations

Let $\Omega \subset \mathbb{R}^2$ be a domain, given a smooth function u defined on Ω, we consider the **second order quasilinear operator** of the form

$$\sum_{i,j=1}^{2} a_{ij}(x, u, \nabla u)\partial_{ij}^2 u + b(x, u, \nabla u) = 0, \tag{2.5}$$

where $x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2$ and a_{ij} and b are C^1 functions defined on $\Omega \times \mathbb{R} \times \mathbb{R}^2$, i.e., $a_{ij}, b \in C^1(\Omega \times \mathbb{R} \times \mathbb{R}^2)$. Here ∇u denotes the gradient of u, that is, $\nabla u = (\partial_1 u, \partial_2 u)$. Moreover, we ask for the matrix $A = (a_{ij})_{i,j=1,2}$ to be symmetric.

Unless otherwise stated, it will be assumed that $u \in C^\infty(\Omega)$. Nevertheless, the differentiability hypothesis assumed here are much more restrictive than it really requires, $u \in C^2$ is enough (see [10, Chapter 10]).

Remark 3:
One can consider more general type of elliptic equations, the ones of fully nonlinear type (see [10, Chapter 17] for related results on this matter). Nevertheless, we pretend to apply the Maximum Principle for Constant Mean Curvature surfaces in Chapter 3, for which second order quasilinear operators apply.

We can represent (2.5) as

$$Qu = 0,$$

where Q is the differential operator defined by

$$Qu \equiv \sum_{i,j=1}^{2} a_{ij}(x, u, \nabla u)\partial_{ij}^2 u + b(x, u\nabla u) \tag{2.6}$$

As we did for the linear case, we adopt the following definitions:

Definition 2.7:
Let Q be a differential operator as in (2.6)*. Then, we say Q is*

- ***Elliptic*** *if all the eigenvalues of the matrix A are positive.*
- ***Uniformly Elliptic*** *if the eigenvalues of the matrix A are bounded below and above by a positive constant.*
- ***Hyperbolic*** *if all the eigenvalues of A are non-vanishing, but there exist positive and negative eigenvalues.*
- ***Parabolic*** *if some eigenvalues vanishes.*

Let us denote by

$$m(x,p,q) = \min\{a_{ij}(x,t,p) \,:\, i,j=1,2\}$$

and

$$M(x,p,q) = \max\{a_{ij}(x,t,p) \,:\, i,j=1,2\},$$

for $(x,t,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^2$. Then, ellipticity means that

$$0 < m(x,t,p)|\xi|^2 \leq \sum_{i,j=1}^{2} a_{ij}(x,t,p)\xi_i\xi_j \leq M(x,t,p)|\xi|^2, \tag{2.7}$$

for all $(x,t,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^2$, where $\xi = (\xi_1, \xi_2)$ and $|\cdot|$ denotes the usual Euclidean norm. Moreover, if M/m is bounded then L is uniformly elliptic.

Define the scalar function

$$T(x,t,p) = \sum_{i,j=1}^{2} a_{ij}(x,t,p)p_ip_j, \tag{2.8}$$

for all $(x,t,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^2$. So, we can express (2.7) as

$$0 < m(x,t,p)|p|^2 \leq T(x,t,p) \leq M(x,t,p)|p|^2, \tag{2.9}$$

for all $(x,t,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^2$.

There are two types of differential operator that are worth to recall. They are particular cases of quasilinear second order differential operator and they appear in most of the geometric problems we consider.

2.2.1 Divergence form

The operator Q, given by (2.6), is of **divergence form** if there exists a differentiable function $A(x,t,p) = (A_1(x,t,p), A_2(x,t,p))$ and a scalar function $B(x,t,p)$ such that

$$Qu = \operatorname{div}(A(x,u,\nabla u)) + B(x,u,\nabla u),\ u \in C^2(\Omega). \tag{2.10}$$

Note that a differential operator Q of divergence form is a second order quasilinear differential operator for

$$a_{ij}(x,t,p) = \frac{1}{2}\left(\partial_{p_i} A_j(x,t,p) + \partial_{p_j} A_i(x,t,p)\right).$$

But the reciprocal is not always true, a second order quasilinear operator with smooth coefficients is not necessary expressible in divergence form.

For example, the function u that defines a minimal graph in the Eucliean space satifies $Mu = 0$, where M is of divergence form given by

$$Mu = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right). \tag{2.11}$$

One we can also compute

$$T(x,t,p) = \frac{|p|^2}{1+|p|^2},$$

therefore the equation is elliptic.

2.2.2 Variational

The operator Q is **variational** if it is the Euler-Langrange operator corresponding to a multiple integral

$$\int_\Omega F(x,u,\nabla x)\,dx,$$

where F is a differentiable scalar function. In fact, a variational operator is of divergence form for

$$A_i(x,t,p) = \partial_{p_i} F(x,t,p) \text{ and } B(x,t,p) = -\partial_t F(x,t,p),$$

the converse, obviously, is not always true.

In this case, ellipticity of Q is equivalent to the strict convexity of the function F with respect to the p variables.

The minimal surface equation appears also from a variational problem, they are critical points for the area functional, that is, Mu given by (2.11) is equivalent to the variational operator associated with the integral

$$\int_\Omega (1+|\nabla u|^2)^{1/2}\,dx.$$

2.2.3 Tangency Principle for quasilinear operators

The idea for proving the Maximum Principle for quasilinear operators is to reach the linear case already developed. To do so, we need the following.

Lemma 2.8 (Hadamard Lemma)**:**
Let $U \subset \mathbb{R}^n$ be a convex domain and $f : U \to \mathbb{R}$ a C^1 function. Then

$$f(x) - f(y) = \sum_{i=1}^{n} h_i(x, y)(x_i - y_i),$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ and

$$h_i(x, y) = \int_0^1 \partial_i f(sx + (1 - s)y)\, ds,$$

for all $1 \leq i \leq n$ and $x, y \in U$.

Proof. Consider the function $g(s) = f(sx+(1-s)y)$ for $s \in [0, 1]$. Therefore, by the Fundamental Theorem of Calculus, we obtain

$$\begin{aligned} f(x) - f(y) &= \int_0^1 g'(s)\, ds = \int_0^1 \sum_{i=1}^{n} \partial_i f(sx + (1 - s)y)\ (x_i - y_i)\ ds \\ &= \sum_{i=1}^{n} \left(\int_0^1 \partial_i f(sx + (1 - s)y)\, ds \right) (x_i - y_i) \\ &= \sum_{i=1}^{n} h_i(x, y)\, (x_i - y_i)\,. \end{aligned}$$

□

This simple calculus lemma allows us to fall into the linear case.

Lemma 2.9:
Let $u^k : \Omega \to \mathbb{R}$, $k = 1, 2$, be two solutions to the same quasilinear equation

$$Q(u^k) = \sum_{i,j=1}^{2} a_{ij}(x, u^k, \nabla u^k)\partial_{ij}^2 u^k + b(x, u^k, \nabla u^k) = 0,$$

for $k = 1, 2$, in a domain $\Omega \subset \mathbb{R}^2$. Then, $u := u^1 - u^2$ satisfies an elliptic linear equation $Lu = 0$.

Proof. Let us denote

$$q^k = \left(\partial^2_{ij} u^k\right)_{i,j=1,2} \text{ and } p^k = (\partial_i u^k)_{i=1,2} \text{ for } k = 1, 2.$$

Set $q := (q_{ij})_{i,j=1,2} \in \mathbb{R}^4$, $p = (p_i)_{i=1,2} \in \mathbb{R}^2$, $u \in \mathbb{R}$ and $x = (x_i)_{i=1,2} \in \Omega$, and define

$$\phi(q, p, u, x) = \sum_{i,j=1}^{2} a_{ij}(x, u, p) q_{ij} + b(x, u, p).$$

Since $Q(u^k) = 0$, we have

$$\phi(q^1, p^1, u^1, x) - \phi(q^2, p^2, u^2, x) = 0,$$

and using Hadamard Lemma we get

$$\sum_{i,j=1}^{2} A_{ij}(x)(\partial^2_{ij} u^1 - \partial^2_{ij} u^2) + \sum_{i=1}^{2} B_i(x)(\partial_i u^1 - \partial_i u^2) + C(x)(u^1 - u^2) = 0,$$

where

$$A_{ij}(x) = \int_0^1 a_{ij}(x, \left(su^1 + (1-s)u^2\right)(x), \left(s\nabla u^1 + (1-s)\nabla u^2\right)(x))\, ds,$$

or equivalently,

$$Lu = 0,$$

where $u = u^1 - u^2$ and

$$Lu = \sum_{i,j=1}^{2} A_{ij}(x)\partial^2_{ij} u + \sum_{i=1}^{2} B_i(x)\partial_i u + C(x)u.$$

Finally, since $(a_{ij})_{ij=1,2}$ is definite positive, $(A_{ij})_{i,j=1,2}$ is definite positive for all $x \in \Omega$, therefore L is an elliptic linear operator. □

Now, we are ready to announce

Theorem 2.10 (Tangency Principle)**:**
Let $u^k : \Omega \to \mathbb{R}$, $k = 1, 2$, be two C^2 function defined on an open domain Ω of either $\mathbb{R}^2$ or the halfspace $\left\{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0\right\}$ such that $(0, 0) \in \Omega$.

Assume u^k, $k = 1, 2$, is a solution to the same quasilinear elliptic equation $Q(u^k) = 0$. Then,

- *if $(0, 0)$ is an interior point at Ω, $u^1(0, 0) = u^2(0, 0)$ and $u^1 \leq u^2$ in Ω, then $u^1 \equiv u^2$ in Ω,*

- *if* $(0,0)$ *is a boundary point at* Ω, $u^1(0,0) = u^2(0,0)$, $\partial_2 u^1(0,0) = \partial_2 u^2(0,0)$ *and* $u^1 \leq u^2$ *in* U, *then* $u^1 \equiv u^2$ *in* Ω.

Proof. We just need to apply Lemma 2.9 to the difference $u = u^1 - u^2$. Now, the result follows from Corollary 2.6. □

Chapter 3

Surfaces of Constant Mean Curvature

In this part we will use the Maximum Principle for geometric applications. Using the geometric version of the Maximum Principle we will be able to classify properly embedded constant mean curvature surfaces, $H \neq 0$, in the Euclidean Space following the ideas of Korevaar-Kusner-Meeks-Solomon [13, 14]. In fact, we extend the classification results to families of surfaces that satisfies the Geometric Maximum Principle and contains a compact embedded surface in the family (cf. [1]). These are the two main ingredients for proving the classification result in the case of constant mean curvature.

To do so, we first rewrite the Maximum Principle in geometric terms. This allows us to compare surfaces. Once with this tool in hands, we will develop the Alexandrov Reflection Method. Such a method is the fundamental idea to prove that a compact embbeded constant mean curvature surface $H \neq 0$ must be a sphere. We continue by giving geometric height estimates for graphs by using the Geometric Maximum Principle and the existence of a compact embbeded surface, which must be a round sphere, in the family.

Finally, the Alexandrov Reflection Method and the Geometric Height Estimates are the key ingredients to prove the classification result for properly embedded constant mean curvature $H \neq 0$ surfaces in $\mathbb{R}^3$ of finite topology and no more than 2 ends. We can prove

- With 0 ends, it must be a round sphere (Alexandrov‘s Theorem).
- There is no properly embedded constant mean curvature surfaces in $\mathbb{R}^3$ with finite topology and one end.
- With 2 ends, it must be a Delaunay surface.

3.1 Preliminaries and Notation

As a first step in this chapter, we present the space form model that we work with and the notation for surfaces we will use (see [8] or [18]).

Although the results we will establish are for surfaces in a three-dimensional Euclidean Space, we will do the $(n+1)-$dimensional description of them with the same amount of energy. Denote by $\mathbb{R}^{n+1}$ the Euclidean $(n+1)-$dimensional space, that is, the complete, simply connected Riemannian manifold with constant sectional curvature 0, and dimension $n+1$, $n \geq 1$. The usual Euclidean metric is given by

$$\langle\ ,\ \rangle = \sum_{i=1}^{n+1} dx_i^2\,.$$

The group of isometries of $\mathbb{R}^{n+1}$ is given by the semi-direct product of $\mathcal{O}(n+1)$ and $\mathcal{T}(n+1)$, where $\mathcal{O}(n+1)$ the orthogonal group generated by all the linear maps of $\mathbb{R}^{n+1}$ that preserve the scalar product, and $\mathcal{T}(n+1)$ is the group of affine traslations of $\mathbb{R}^{n+1}$.

Then, the totally geodesics submanifolds of $\mathbb{R}^{n+1}$ are the affine subspaces. In particular, the geodesics in $\mathbb{R}^{n+1}$ passing through $p \in \mathbb{R}^{n+1}$ with velocity $v \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$ is given by

$$\gamma_{(p,v)}(t) = p + tv\,,$$

where t is the arc length parameter of γ.

3.1.1 Surfaces in the Euclidean Space

Let $\Sigma \subset \mathbb{R}^3$ be an oriented surface. The unit normal vector field N along the surface will be called **Gauss map**. We define the **Second Fundamental Form** of the immersion associated to N at a point $p \in \Sigma$, as

$$II(u,v) = \langle -\nabla_u N, v\rangle\,,$$

where $u, v \in T_p\Sigma$ are tangent vectors to Σ at p, $I \equiv \langle\ ,\ \rangle$ is the **First Fundamental Form of** Σ and $\overline{\nabla}$ is the Levi-Civita connection on Σ.

The **shape operator** of the surface is given by

$$S(v) = -\overline{\nabla}_v N,\ \ v \in T_p\Sigma\,.$$

Let k_1 and k_2 be the principal curvatures of Σ associated to N; that is, the eigenvalues of the shape operator. The **mean curvature** H and the **extrinsic curvature** K_e of Σ are defined as $2H = k_1 + k_2$ and $K_e = k_1 k_2$, respectively.

Denote by K the **Gaussian curvature**, or intrinsic curvature, of the First Fundamental Form of Σ. The Gauss equation relates the extrinsic and the intrinsic curvature of a surface $\Sigma \subset \mathbb{R}^3$, that is,

$$K = K_e. \tag{3.1}$$

It is well known that the coefficients of the First and Second Fundamental Forms of a immersed surface satisfy the Codazzi equations [8], in other words,

$$\nabla_X SY - \nabla_Y SX - S[X,Y] \equiv 0,\ X, Y \in \mathfrak{X}(X). \tag{3.2}$$

3.2 Geometric Maximum Principle

We want to use the Maximum Principle developed in Chapter 2 for comparing surfaces. To do so, we will consider a surface $\Sigma \subset \mathbb{R}^3$ (locally) as a graph on a domain of its tangent plane, that is,

$$\Sigma = graph(u) := \{p + u(p)n(p),\ p \in \Omega\},$$

where $u \in C^2(\Omega)$, $\Omega \subset T_p\Sigma \equiv \mathbb{R}^2$ is a domain in the tangent plane to Σ, n the normal vector field along the tangent plane.

Definition 3.1:
We say that $\Sigma_i = graph(u_i)$, $i = 1, 2$ satisfy the Maximum Principle if the difference $\omega = u_1 - u_2$ verifies a differential equation $L\omega \geq 0$ that satisfies the conclusion of the Tangency Principle (Theorem 2.10). Here, L is a differential operator invariant under the isometries of $\mathbb{R}^3$.

Note that a large amount of families of surfaces verify Tangency Principle. Classical examples of this fact are the family of surfaces with constant mean curvature, in short $H-$surfaces, and the family of surfaces with positive constant extrinsic curvature K_e, in short, K_e-sufaces. And, more generally, the family of special Weingarten surfaces in $\mathbb{R}^3$ satisfying a relation of the type $H = f(H^2 - K_e)$, where f is a differentiable function defined on an interval $\mathcal{J} \subseteq [0, \infty)$ with $0 \in \mathcal{J}$, such that $4tf'(t)^2 < 1$ for all $t \in \mathcal{J}$ (see [17]).

Remark 4:
In order to extend some local proprieties, we will assume, if necessary, that the surfaces are analytic. Observe that this hypotheses is satisfied if L is an uniformly elliptic operator with analytic coefficients and $u \in C^2(\Omega)$. In particular, constant mean curvature surfaces satisfy such condition.

Let us see this in the case of a $H-$surface. Let $\Sigma = \text{graph}(u)$ be the graph of a function $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$. A straightforward computation shows that the normal along Σ is given by

$$N := \frac{1}{\sqrt{1 + |\nabla u|^2}}(-\nabla u, 1),$$

and therefore

$$\text{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = nH.$$

Therefore, from (2.11), one can see that u satisfies

$$\tilde{M}u = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) - nH = 0, \tag{3.3}$$

which is an elliptic operator on Divergence form.

Now, we focus on the geometric interpretation of a tangency point. For that, we need the following definitions. First, the geometric meaning of an interior tangent point.

Definition 3.2:
Let $\Sigma_i \subset \mathbb{R}^3$ be surfaces, $i = 1, 2$, and $p \in \operatorname{int}(\Sigma_1) \cap \operatorname{int}(\Sigma_2)$ such that the tangent plane and the normal unit vector of Σ_1 and Σ_2 at p coincide. In these conditions, we say that p is an ***interior tangent point of*** Σ_1 ***and*** Σ_2 *(see Figure 3.1).*

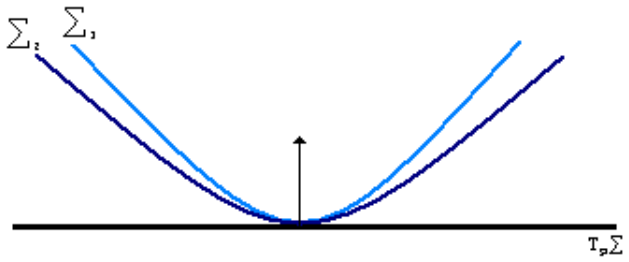


Figure 3.1: Interior tangent point.

Second, the geometrical meaning of a boundary tangent point.

Definition 3.3:
Let $\Sigma_i \subset \mathbb{R}^3$ be surfaces with smooth $\partial\Sigma_i \neq \emptyset$, $i = 1, 2$. Let $p \in \partial\Sigma_1 \cap \partial\Sigma_2$ be a point such that the tangent planes and the unit normal vector of Σ_1 and Σ_2, and the interior conormal vectors of $\partial\Sigma_1$ and $\partial\Sigma_2$, coincide at p. In this conditions, we say that p is a ***boundary tangent point of*** Σ_1 ***and*** Σ_2.

Observe that if p is either interior or boundary tangent point of $\Sigma_1 = graph(u_1)$ and $\Sigma_2 = graph(u_2)$, then u_1 and u_2 are defined over the same tangent plane. In this case, we say that

Σ_1 is **above** (resp. **below**) of Σ_2, if $u_1 \geq u_2$ (resp. $u_1 \leq u_2$). We denote this by $\Sigma_1 \geq \Sigma_2$ (resp. $\Sigma_1 \leq \Sigma_2$).

So, if Σ_1 is above Σ_2 and both have constant mean curvature, u_1 and u_1 satisfy the same elliptic partial differential equation and $u_1 \geq u_2$ (locally). So, the geometric version of Theorem 2.10 is settled as:

Theorem 3.4 (Geometric Maximum Principle)**:**
Let Σ_i, $i = 1, 2$, be surfaces in $\mathbb{R}^3$ so that satisfy the Maximum Principle (see Definition 3.1). Assume that p is an interior or boundary tangent point of Σ_1 and Σ_2 and, in a neighborhood of p, $\Sigma_1 \geq \Sigma_2$. Then $\Sigma_1 = \Sigma_2$.

Hence, the Geometric Maximum Principle allows us to compare surfaces. It is usual to compare a surface to another one in the family that we already know, for example, either a revolution or an umbilical surface, or with the reflection of the surface itself.

3.3 Alexandrov Reflection Method for compact domains

Henceforth in this chapter, we will work on the Euclidean 3−space, but it is easy to realize that we can extend these methods to the Hyperbolic 3−space or to a hemisphere on $\mathbb{S}^3$.

The method known as Alexandrov reflection method (or technique of moving planes) is an important consequence of the Maximum Principle. In 1956, Alexandrov [2] showed that every compact, embedded surface with (non vanishing) constant mean curvature in $\mathbb{R}^3$ should be a round sphere. This result is known as Alexandrov Theorem. The proof consists in comparing the surface to its reflection with respect to planes. Recall that reflections with respect to totally geodesic planes are isometries in $\mathbb{R}^3$. In $\mathbb{H}^3$ or a hemisphere of $\mathbb{S}^3$ we reflect with respect to totally geodesic surfaces and again, these reflections are isometries.

We will work with families of surfaces satisfying the Maximum Principle (in the sense of Definition 3.1), for example, the family of surfaces of $H-$surfaces.

Definition 3.5:
We say that a family $\mathcal{F}$ of oriented surfaces in $\mathbb{R}^3$ satisfies the **Hopf Maximum Principle** *if the following properties are satisfied:*

1. *$\mathcal{F}$ is invariant under isometries of $\mathbb{R}^3$. In other words, if $\Sigma \in \mathcal{F}$ and φ is an isometry of $\mathbb{R}^3$, then $\varphi(\Sigma) \in \mathcal{F}$.*
2. *If $\Sigma \in \mathcal{F}$ and $\widetilde{\Sigma}$ is another surface contained in Σ, then $\widetilde{\Sigma} \in \mathcal{F}$.*
3. *There is an embedded compact surface without boundary in $\mathcal{F}$.*
4. *Any two surfaces in $\mathcal{F}$ satisfy the Geometric Maximum Principle (interior and boundary).*

As it was pointed out above, the family of special Weingarten surfaces in $\mathbb{R}^3$ is included in the above definition. We also should remark now that if a family of surfaces $\mathcal{F}$ verifies the Hopf Maximum Principle, then there exists, up to isometries, an unique embedded compact surface without boundary. Such a surface is, necessary, a totally umbilical sphere (see Theorem 3.7 below). In particular, for the family of $H-$surfaces with $H \neq 0$, this is the Alexandrov Theorem.

Beforehand, we shall establish some previous definitions.

Definition 3.6:
Let $P \subset \mathbb{R}^3$ be a totally geodesic plane with unit normal vector field N_P. We define the **oriented foliation of planes associated to** *P, denoted by $P(t)$, to be the parallel planes P at distance t (see Figure (3.2)), that is,*

$$P(t) = P + tN_P.$$

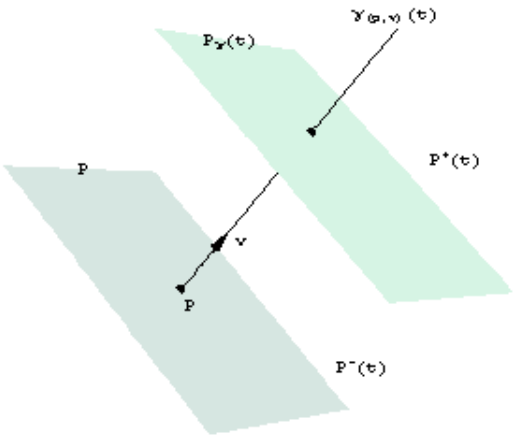


Figure 3.2: Parallel planes.

Also, we will use the notation $P^-(t)$ and $P^+(t)$ for referring the half spaces determined by $P(t)$, i.e.,

$$P^-(t) = \bigcup_{s \leq t} P(s)\,, \tag{3.4}$$

$$P^+(t) = \bigcup_{s \geq t} P(s)\,. \tag{3.5}$$

For a subset $G \subset \mathbb{R}^3$, let

$$G^+(t) = G \cap P^+(t)\,, \quad G^-(t) = G \cap P^-(t)\,.$$

Moreover, $\widetilde{G}^+(t)$ represents the reflection of $G^+(t)$ with respect to $P(t)$ and, analogously, $\widetilde{G}^-(t)$ is the reflection of $G^-(t)$ with respect to $P(t)$.

3.3.1 Alexandrov Theorem

Now, we are ready to prove:

Theorem 3.7 (Alexandrov Theorem)**:**
Let $\Sigma \in \mathcal{F}$ be embedded compact surface without boundary. Here $\mathcal{F}$ is a family of oriented surfaces that verifies the Hopf Maximum Principle (see Definition 3.5). Then, Σ is a totally umbilical sphere. Moreover, up to isometries, such a sphere is unique.

Proof. Since $\Sigma \subset \mathbb{R}^3$ is a connected and embedded compact surface, Σ is orientable and it separates $\mathbb{R}^3$ in two connected components. Set $\mathcal{W}$ the compact region of $\mathbb{R}^3$ with boundary Σ.

Let P be a totally geodesic plane disjoint from $\mathcal{W}$, it is clear that we can find such P from compactness of $\mathcal{W}$, and $P(t)$ be the oriented foliation of planes associated to P so that $\mathcal{W} \subset P^-$. Now, move $P(t)$ towards $\mathcal{W}$, that is, decreasing t until $P(t)$ touches Σ at a first point q, say $t = t_0$. In fact, the intersection might be more than one point, it must be a compact set. We will assume that it is only one point for simplicity.

Then, since Σ has bounded curvature, there exists $\epsilon > 0$ so that $\Sigma^+(t)$ is a graph of bounded slope over a domain of $P(t)$. Moreover, shrinking $\epsilon > 0$ if necessary, $\mathrm{int}(\tilde{\Sigma}^+(t)) \subset \mathcal{W}$ for all $t \in (t_0 - \epsilon, t_0)$, since the extrinsic curvature is nonnegative at q. In fact, at any connected component which is not a point it must have nonnegative extrinsic curvature at each point of the intersection. The important fact here is that $\Sigma^-(t)$ is convex for $t \in (t_0 - \epsilon, t_0)$ and ϵ small enough.

Furthermore, the normal vector field at any point of $\tilde{\Sigma}^+(t)$ is the reflection of the normal vector field at the corresponding point of $\Sigma^+(t)$. We continue decreasing t till the first τ where one of the following conditions fails to hold:

(a) $\mathrm{int}(\tilde{\Sigma}^+(\tau)) \subset \mathcal{W}$.

(b) $\Sigma^+(\tau)$ is a graph of bounded slope over a domain of $P(\tau)$.

If (a) fails, we apply the Geometric Maximum Principle (Theorem 3.4) to $\Sigma^-(\tau)$ and $\tilde{\Sigma}^+(\tau)$ at the point where they touch to conclude that $P(\tau)$ is a plane of symmetry of Σ. If (b) fails first, then the point p where the tangent space of $\Sigma^+(\tau)$ becomes orthogonal to $P(\tau)$ belongs to $\partial\Sigma^+(\tau) = \partial\Sigma^-(\tau) \subset P(\tau)$, i.e., $\Sigma^-(\tau)$ and $\tilde{\Sigma}^+(\tau)$ has a boundary tangent point. Again, by the Geometric Maximum Principle, $P(\tau)$ is a plane of symmetry of Σ.

Thus, for any direction, one finds a plane of symmetry of Σ and so, from [12, Lemma 2.2, Chapter VII], Σ is a totally umbilical sphere.

In addition, there cannot be two totally umbilical spheres Σ_1, Σ_2 in $\mathcal{F}$ which are non isometric. Otherwise, up to isometries, we can suppose that one of them, let us say Σ_1, is contained in the bounded region determined by Σ_2. If we move Σ_1 until it meets first Σ_2 then, at this contact point, the normal vectors to Σ_1 and Σ_2 coincide and we can conclude that $\Sigma_1 = \Sigma_2$ by the Maximum Principle. If the normal vectors at that point do not coincide, we keep on moving Σ_1 such that its center runs along a half-line, until it meets Σ_2 at the last contact point. At that contact point the normal vectors do necessarily coincide, which allows us, as before, to assert that $\Sigma_1 = \Sigma_2$. □

3.4 Height Estimates

Our aim in this section is to develop a geometric method for obtaining an upper bound to the maximum height that an embedded compact surface in $\mathbb{R}^3$ can achieve. The estimates provided here are not optimal, but more than enough for giving interesting consequences on embedded surfaces (see [1, 13, 14, 15, 17]).

We shall start studying graphs Σ with boundary contained in a plane P of $\mathbb{R}^3$. Up to an isometry, we can assume that P is the $xy-$plane, and so

$$\Sigma = \left\{(x, y, u(x, y)) \in \mathbb{R}^3 : \ (x, y) \in \Omega \subseteq \mathbb{R}^2\right\}.$$

Theorem 3.8:
Let $\mathcal{F}$ be a family of surfaces in $\mathbb{R}^3$ satisfying the Hopf Maximum Principle. Let $\Sigma \in \mathcal{F}$ be a compact graph on a domain Ω in the $xy-$plane with $\partial\Sigma$ contained in this plane. Then for all $p \in \Sigma$, the distance in $\mathbb{R}^3$ from p to the $xy-$plane is less or equal to $4R_{\mathcal{F}}$. Here, $R_{\mathcal{F}}$ stands for the radius of the unique totally umbilical sphere in the family $\mathcal{F}$.

Proof. Let $\Sigma \in \mathcal{F}$ be a graph on a domain Ω in the $xy-$plane and Σ_0 the unique totally umbilical sphere of $\mathcal{F}$. Let $P(t)$ be the foliation of $\mathbb{R}^3$ by horizontal planes, where $P(t)$ is the plane at height t.

> **Claim 1:** *For every $t > 2R_{\mathcal{F}}$, the diameter of any open connected component bounded by $\Sigma(t) = P(t) \cap \Sigma$ is less than or equal to $2R_{\mathcal{F}}$.*

Proof of Claim 1: Indeed, let us suppose that this claim is not true. Then, for some connected component $C(t)$ of $\Sigma(t)$, there are points p, q in the interior of the domain $\Omega(t)$ in $P(t)$ bounded by $C(t)$ such that $\text{dist}(p, q) > 2R_{\mathcal{F}}$. Let Q be the domain in $\mathbb{R}^3$ bounded by $\Sigma \cup \Omega$. Let β be a curve in $\Omega(t)$ joining p and q, β and $C(t)$ being disjoint. Let Π be the *rectangle* given by

$$\Pi = \{\alpha_s(r) \ : \ s \in \mathcal{I}, \, r \in [0, t]\}$$

where $\mathcal{I}$ is the interval where β is defined, and α_s is the geodesic with initial data $\alpha_s(0) = \beta(s)$ and $\alpha_s'(0) = -e_3$, r being the length arc parameter along α_s and $e_3 = (0, 0, 1)$.

Since Σ is a geodesic graph and β is contained in the interior of the domain determined by $C(t)$, then $\Pi \subset Q$. Let $\widetilde{p} \in \Pi$ be a point whose distance to $\partial\Pi$ is greater than $R_{\mathcal{F}}$. Note that, according to our construction of Π, the point $\widetilde{p}$ necessarily exists.

Let $\eta(r)$ be a horizontal geodesic passing through $\widetilde{p}$ and such that every point in $\eta(r)$ is far from $\partial\Pi$ a distance greater than $R_{\mathcal{F}}$. Observe that such a geodesic can be chosen as the horizontal line in the plane $P(t_1)$ containing the point $\widetilde{p}$ and being orthogonal to the vector joining p and q. Let $\widetilde{q_1}$ be the first point where η meets Q, and $\widetilde{q_2}$ the last one.

Now, let us consider the spheres $\Sigma_0(r) \in \mathcal{F}$ centered at every point $\eta(r)$. Note that these spheres can be obtained from the rotational sphere Σ_0 by means of a translation of $\mathbb{R}^3$.

There exists a first sphere in this family (coming from $\widetilde{q_1}$) which meets Σ. If the normal vectors of both surfaces coincide at this point, we conclude that both surfaces agree by the maximum principle. If the normal vectors are opposite, we reason as follows.

Let us consider the first sphere $\Sigma_0(r_0)$ in the family above (coming from $\widetilde{q_1}$) which meets Π at an interior point of Π.

For every $r > r_0$ we consider the piece $\widetilde{\Sigma}_0(r)$ of the sphere $\Sigma_0(r)$ which has gone through Π. Since these spheres leave Q at $\widetilde{q_2}$ and none of them meets $\partial\Pi$, there exists a first value r_1 such that $\widetilde{\Sigma}_0(r_1)$ meets first $\partial Q \cap \Sigma$ at a point $\widetilde{q_0}$. Thus, applying the maximum principle to $\Sigma_0(r_1)$ and Σ at $\widetilde{q_0}$, we conclude that both surfaces agree, which is a contradiction (see Figure (3.3)).

Therefore we obtain that, for height $t = 2R_{\mathcal{F}}$, the diameter of every open connected component bounded by $\Sigma(t) = P(t) \cap \Sigma$ is less than or equal to $2R_{\mathcal{F}}$. This proves Claim 1.

To finish, we will see that $P(t) \cap \Sigma$ is empty for $t > 6R_{\mathcal{F}}$. To do that, it suffices to prove the following assertion

> **Claim 2:** *Let Ω_1 be a connected component bounded by $\Sigma(2R_{\mathcal{F}})$ in $P(2R_{\mathcal{F}})$. The distance from any point in Σ (which is a graph on Ω_1) to the plane $P(2R_{\mathcal{F}})$ is less than or equal to $4R_{\mathcal{F}}$.*

Proof of Claim 2: Let σ be a support line of $\partial\Omega_1$ in $P(2R_{\mathcal{F}})$ with exterior unitary normal vector v. Let $\sigma(4R_{\mathcal{F}})$ be the vertical translation of σ at height $4R_{\mathcal{F}}$, i.e., $\sigma(4R_{\mathcal{F}}) = \sigma + 4R_{\mathcal{F}}e_3$. Let us take $\eta(r)$ a geodesic such that $\eta(0) \in \sigma(4R_{\mathcal{F}})$ and $\eta'(0) = \frac{1}{\sqrt{2}}(v + e_3)$.

Now, let us consider for every r the plane $\Pi(r)$ in $\mathbb{R}^3$ passing through $\eta(r)$ which is orthogonal to $\eta'(r)$. Such planes intersect every horizontal plane in a line parallel to σ, being $\pi/4$ the angle between them.

If the Claim 2 was not true, there would exist a point $p \in \Sigma$ over Ω_1 such that its height on the plane $P(2R_{\mathcal{F}})$ would be greater than $4R_{\mathcal{F}}$.

Let Σ_1 be the compact piece of Σ which is a graph on Ω_1. Observe that:

(i) For r big enough, $\Pi(r)$ does not meet Σ_1, that is, there exists r (big enough) so that $\Sigma_1 \subset \Pi^-(r)$.

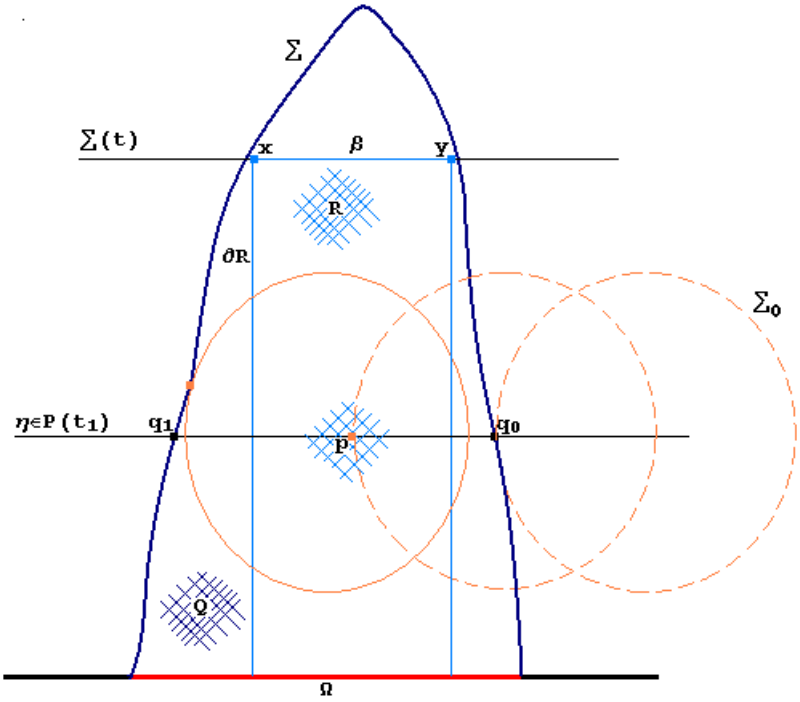


Figure 3.3: Σ_0 passing through Π.

(ii) For $r = 0$ the plane $\Pi(0)$ contains the line $\sigma(4R_{\mathcal{F}})$, $p \in \Pi^+(0)$ and the reflection of p with respect to $\Pi(0)$ is a point whose vertical projection on $P(2R_{\mathcal{F}})$ is not in Ω_1 since its height on the plane $P(2R_{\mathcal{F}})$ is greater than $4R_{\mathcal{F}}$.

(iii) for $r \geq 0$, $\tilde{\Sigma}_1^+(r)$, that is, the reflection with respect to $\Pi(r)$ of the part of Σ_1 in $\Pi^+(r)$, does not touch $\partial\Omega_1$.

Now, move $\Pi(r)$ towards Σ_1, that is, decreasing r from infinity, until $\Pi(r_0)$ such that it touches Σ_1 at a first point q (we can ensuring this by item (i)). Then, there exists $\epsilon > 0$ so that, for $r \in (r_0 - \epsilon, r_0)$, $\Sigma_1^+(r)$ is a graph of bounded slope over a domain of $\Pi(r)$ and $\text{int}(\tilde{\Sigma}_1^+(r)) \subset \mathcal{W}$ (where $\mathcal{W}$ is the bounded domain of $\mathbb{R}^3$ bounded by $\Sigma_1 \cup \Omega_1$). Furthermore, the normal vector field at any point of $\tilde{\Sigma}_1^+(r)$ is the reflection of the normal vector field at the corresponding point of $\Sigma_1^+(r)$. We continue decreasing r till the first $\bar{r}$ where one of the following conditions fails to hold:

(a) $\text{int}(\tilde{\Sigma}^+(\bar{r})) \subset \mathcal{W}$.

(b) $\Sigma^+(\bar{r})$ is a graph of bounded slope over a domain of $\Pi(\bar{r})$.

Note that one of the above situation should happen by items (ii) and (iii). So, we can get a contradiction as in Theorem 3.7.

Now, If (a) fails, we apply the Geometric Maximum Principle (Theorem 3.4) to $\Sigma_1^-(\bar{r})$ and $\tilde{\Sigma}_1^+(\bar{r})$ at the point where they touch to conclude that $\Pi(\bar{r})$ is a plane of symmetry of Σ_1. If (b) fails first, then the point $\bar{p}$ where the tangent space of $\Sigma_1^+(\bar{r})$ becomes orthogonal to $\Pi(\bar{r})$ belongs to $\partial\Sigma_1^+(\bar{r}) = \partial\Sigma_1^-(\bar{r}) \subset \Pi(\bar{r})$, i.e., $\Sigma_1^-(\bar{r})$ and $\tilde{\Sigma}_1^+(\bar{r})$ has a boundary tangent point. By the Geometric Maximum Principle (Theorem 3.4), $\Pi(\bar{r})$ is a plane of symmetry of Σ_1.

In any case, this is impossible since it would mean that Σ_1 is a compact surface with no boundary. This proves Claim 2.

This finishes the proof. □

Using this result and the Alexandrov Method, we can bound the maximum distance attained by an embedded compact surface whose boundary is contained in a plane.

Corollary 3.9:
Let $\mathcal{F}$ be a family of surfaces in $\mathbb{R}^3$ satisfying the Hopf Maximum Principle. Then every embedded compact surface $\Sigma \in \mathcal{F}$ whose boundary is contained in a plane P verifies that for every $p \in \Sigma$ the distance in $\mathbb{R}^3$ from p to the plane P is less than or equal to $12R_{\mathcal{F}}$. Here, $R_{\mathcal{F}}$ denotes the radius of the unique totally umbilical sphere contained in $\mathcal{F}$.

Proof. Let $p \in \Sigma$ be a point where the maximum distance to P is attained. Such a point exists since Σ is compact.

Let $\{P(t)\}_{t\geq 0}$ be the foliation by horizontal plane, parallel to P, whose distance to P is t, with $P(0) = P$ and $p \in P(h)$, $h > 0$.

Now, do Alexandrov reflection with the planes $P(t)$, starting at $t = r$, and decrease t. For $\frac{h}{2} < t \leq h$ the reflection of $\Sigma \cap P^+(t)$, with respect to $P(t)$, does not touch $\partial\Sigma$, since $\partial\Sigma \subset P$. It means that the reflection of $\Sigma^+(t) = \Sigma \cap P^+(t)$ with respect to $P(t)$ intersects Σ only at $\Sigma \cap P(t)$ and Σ never is orthogonal to $P(s)$, $t \leq s \leq h$. Otherwise, it would exists $\bar{t} \in (\frac{h}{2}, h)$ so that the tangent space of $\Sigma^+(\bar{t})$ becomes orthogonal to $P(\bar{t})$ belongs to $\partial\Sigma^+(\bar{t}) = \partial\Sigma^-(\bar{t}) \subset P(\bar{t})$, i.e., $\Sigma^-(\bar{t})$ and $\tilde{\Sigma}^+(\bar{t})$ has a boundary tangent point. By the Maximum Principle (Corollary 3.4), $P(\bar{t})$ is a plane of symmetry of Σ. But, this is impossible, since it would mean that Σ is a compact surface with no boundary.

Hence, $\Sigma^+\left(\frac{h}{2}\right)$ is a graph on a domain of $P\left(\frac{h}{2}\right)$ and the result is proved by means of Theorem 3.8. □

Remark 5:
It is clear, even we have employed some particular notation of the Euclidean space, that the above result can be extended for surfaces in $\mathbb{H}^3$.

3.5 Properly embedded annulus

The existence of a maximum principle and height estimates for a family $\mathcal{F}$, allows us to extend the theory developed by Korevaar, Kusner, Meeks and Solomon [13, 14, 15] for constant mean curvature surfaces in $\mathbb{R}^3$ and $\mathbb{H}^3$ to our family $\mathcal{F}$. Those techniques were extended to some families of elliptic special Weingarten surfaces, i.e., satisfying $H = f(H^2 - K_e)$, by Rosenberg and Sa Earp (see [17]) and Aledo, Espinar and Gálvez in all the generality (see [1]). The cornerstone for that was that these families satisfy the Hopf maximum principle and height estimates. We follow here [1, 17].

Throughout this section we denote by $\mathcal{A}$ a topological annulus, i.e., $\mathcal{A}$ is homeomorphic to a punctured closed disk of $\mathbb{R}^2$. Moreover, $\psi : \mathcal{A} \to \mathbb{R}^3$ is an immersion such that $\Sigma = \psi(\mathcal{A}) \in \mathcal{F}$ is a properly embedded surface parametrized by $\mathbb{S}^1 \times [0, +\infty)$. Here, as above, $\mathcal{F}$ is a family that verifies the Hopf maximum principle.

We establish the following result due to W. Meeks [15].

Lemma 3.10 (Separation Lemma)**:**
Let $\Sigma \in \mathcal{F}$ be a properly embedded annulus in $\mathbb{R}^3$. Let P_1 and P_2 be geodesic parallel planes so that the distance in $\mathbb{R}^3$ between P_1 and P_2 is strictly greater than $2R_{\mathcal{F}}$. Denote by P_1^+ and P_2^+ the disjoint half-spaces determined by these planes. Then all the connected component of $\Sigma \cap P_1^+$ or of $\Sigma \cap P_2^+$ are compact.

To prove this Lemma, we shall recall a basic concept from low-dimensional topology (see [16]).

Definition 3.11:
*Let γ and δ be two embedded loops in $\mathbb{R}^3$ so that $\gamma \cap \delta = \emptyset$, let $S \subset \mathbb{R}^3$ be an oriented embedded surface transverse to γ with $\partial S = \delta$. The **linking number between γ and δ**, denoted by $link(\gamma, \delta)$, is the number of points in $\gamma \cap S$, counted with a sign depending on the relative orientation at each intersection point.*

Observe that the linking number verifies:

- The linking number does not depend on the surface S considered.
- The linking number is symmetric, i.e.,

$$link(\gamma, \delta) = link(\delta, \gamma)\,,$$

- We say that δ and γ are not linked if there exists a surface S as above so that,

$$\gamma \cap S = \emptyset\,.$$

- As an easy generalization of the above item we have that, if γ and $\widetilde{\gamma}$ are homotopic in $\mathbb{R}^3 - \delta$, then
$$link(\gamma, \delta) = link(\widetilde{\gamma}, \delta)\,.$$

Now, we have the necessary tools for proving the Separation Lemma.

Proof of the Separation Lemma. Suppose the lemma is false, then both components, $\Sigma \cap P_1^+$ and $\Sigma \cap P_1^+$, have proper non-compact arcs. Then, we can get properly embedded arcs $\alpha_i : [0, +\infty) \to \Sigma \cap P_i^+$, $i = 1, 2$. Denote by p_i the point $\alpha_i(0) \notin \partial\Sigma$, $i = 1, 2$.

So, it is possible to get an embedded arc β in Σ joining p_1 and p_2 so that the arc $\delta = \alpha_1 \cup \beta \cup \alpha_2$ bounds a simply connected domain.

Let P be the vertical plane that is halfway between P_1 and P_2, and B be a geodesic ball in $\mathbb{R}^3$ containing β. Moreover, let $C \subset P$ be a circle such that

- $C \cap B = \emptyset$ and the intersection between B and the disk bounded by C in P is not empty;
- the tubular neighborhood T of C, with radius $r > R_{\mathcal{F}}$ is embedded and $T \cap B = \emptyset$.

We will choose r such that T is contained in the strip bounded by P_1 and P_2.

Let $B_1 \subset \mathbb{R}^3$ be a geodesic ball in $\mathbb{R}^3$ containing $B \cup T$. Since Σ is proper, there exist points $x_i \in \alpha_i - B_1$, for $i = 1, 2$, and an arc γ joining x_1 and x_2 embedded in Σ, that verifies the following conditions.

- $\gamma \cap B_1 = \emptyset$.
- If $\rho \subset \delta$ is the sub-arc joining x_1 and x_2, then the loop $\sigma = \rho \cup \gamma$ verifies $link(\sigma, C) = \pm 1$, this means that there exists a homotopic deformation of σ in Σ so that it touches the interior of the disk bounded by C only once.
- σ bounds a compact disk D in Σ.

Therefore, $T \cap D$ contains a disk D_1 such that $\partial D_1 \subset \partial T$ and $link(\partial D_1, C) = \pm 1$.

Consider the universal cover $\widetilde{T}$ of T, $\pi : \widetilde{T} \to T$ and lift the disk D_1 to a compact disk $\widetilde{D}_1 \subset \widetilde{T}$. Topologically T is $\mathbb{D} \times \mathbb{S}^1$ and D_1 is isotopic to some $\mathbb{D} \times \{point\}$. So, $\widetilde{T}$ is topologically $\mathbb{D} \times \mathbb{R}$ and $\widetilde{D}_1$ is isotopic to some $\mathbb{D} \times \{point\}$. In particular, $\widetilde{D}_1$ separates $\widetilde{T}$ in two connected components. Denote by W the component at which points the mean curvature vector.

Let $\widetilde{C}$ be the curve in $\widetilde{T}$ whose projection by π is C. For each point $p \in C$, denote by $\Sigma_0(p)$ the rotationally symmetric surface centered in p given by Alexandrov Theorem (Theorem 3.7). It is clear that, if the radius of C is large enough, $\Sigma_0(p)$ is contained in T, for each $p \in C$. For any $\widetilde{p} \in \widetilde{C}$, denote by $\widetilde{\Sigma}_0(\widetilde{p})$ the compact surface whose projection by π is $\Sigma_0(p)$, with $\pi(\widetilde{p}) = p$.

Now, there exists a point $\widetilde{q} \in \widetilde{C}$ such that $\widetilde{\Sigma}_0(\widetilde{q})$ is contained in W and is disjoint from $\widetilde{D}_1$. Moving $\widetilde{q}$ along $\widetilde{C}$ towards $\widetilde{D}_1$, there will exist a first contact point $\widetilde{q}_1$ where $\widetilde{\Sigma}_0(\widetilde{q}_1)$ and $\widetilde{D}_1$ are tangent. Then $\widetilde{\Sigma}_0(\widetilde{q}_1)$ is contained in W and, at the tangent point, $\widetilde{\Sigma}_0(\widetilde{q}_1)$ and $\widetilde{D}_1$ have the

same normal vector. Then, by the Geometric Maximum Principle, they should, but this is a contradiction, and the Lemma is proved. □

Now, using this result we establish the cylindrical bound for the properly embedded annulus. Nevertheless, we first need some notations.

Given a point $p \in \mathbb{R}^3$, a direction $v \in \ s^2$ and a real number $R > 0$, denote by $D(p, v, R)$ the disk centered at p and radius R contained in the plane $\left\{q \in \mathbb{R}^3 : \langle q - p, v\rangle = 0\right\}$. Denote by $C(p, v, R)$ the (solid) cylinder centered at p with radius R in the direction of v, i.e.,

$$C(p, v, R) = \{y + sv \in \mathbb{R}^3;\, s \in \mathbb{R},\, y \in D(p, v, R)\}\,,$$

and

$$C^+(p, v, R) = \{y + sv \in \mathbb{R}^3;\, s > 0,\, y \in D(p, R)\},$$

is the (solid) half-cylinder centered at p with radius R in the direction of v.

Definition 3.12:
*A unit vector $v \in \mathbb{S}^2$ is an **axial vector** for $\Sigma \subset \mathbb{R}^3$ if there exists a sequence of points $\{p_n\} \subset \Sigma$ so that $|p_n| \to +\infty$ and $\frac{p_n}{|p_n|} \to v$.*

Remark 6:
We can have analogous definitions in the Hyperbolic Space. A solid cylinder is nothing but the set of points at a fixed distance from a geodesic.

Now, we are ready to prove.

Theorem 3.13 (Cylindrically bounded)**:**
Let $\psi : \mathcal{A} \to \mathbb{R}^3$ be a properly embedded annulus with $\Sigma = \psi(\mathcal{A}) \in \mathcal{F}$, where $\mathcal{F}$ is a family of surfaces in $\mathbb{R}^3$ satisfying the Hopf maximum principle. Then there exists an axial vector v and a radius $R < \infty$ such that $\Sigma \subset C(O, v, R)$, where O is the origin of $\mathbb{R}^3$ (see Figure 3.4).

Proof. Let $p_n \in \Sigma$ a sequence of points such that $|p_n| \to +\infty$ and $\frac{p_n}{|p_n|} \to v$ as $n \to +\infty$. This means that v is an axial vector. It is clear that such vector exists since $\frac{p_n}{|p_n|} \in \mathbb{S}^2$, which is compact.

We may suppose, without lost of generality, that such axial vector of Σ is e_3. Let B be a geodesic ball of radius $r > 8R_{\mathcal{F}}$ containing the boundary of the annulus, i.e. $\partial\Sigma \subset B$.

Let P be a plane parallel to the e_3-axis and disjoint from B so that $B \subset P^+$, let P_ϵ be a plane in $\mathbb{R}^3$ such that $B \cap P_\epsilon = \emptyset$ and the angle between P and P_ϵ is ϵ. Specifically, if w is the unit normal along P pointing at P^+, then $w_\epsilon := w + \epsilon e_3$ is the unit normal along P_ϵ.

Since $\text{dist}(p_n, P_\epsilon) \to +\infty$ as $n \to +\infty$, the Separation Lemma (Lemma 3.10) and the Height Estimates (Theorem 3.9) imply that each connected components of Σ contained in the half-space

determined by P_ϵ disjoint from B are compact. Again, by the Height Estimates, these compact components has bounded distance from P_ϵ.

Letting $\epsilon \to 0$, we conclude that the components of Σ in the half-space determined by P disjoint from B, are a uniformly bounded distance from P. Hence, moving P up this fixed distance C, we obtain that Σ is completely contained in the half-space determined by P containing B. But, we can argue as above for any parallel plane to the axis, and the distance C is the same for all the planes. So, Σ is cylindrical bounded. □

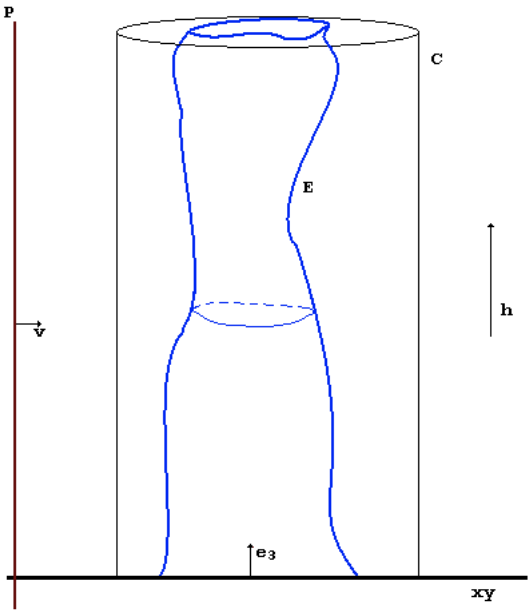


Figure 3.4: Cylindrically bounded.

The theorem above asserts that for any properly embedded annulus there exists a unique axial vector. In addition, this vector is the generator of the rulings of the cylinder.

3.6 Alexandrov Reflection Method for non-compact domains

The aim now, is to extend the Alexandrov reflection method to non-compact surfaces. The problem is that the first contact point could be *at infinity*. We follow [13] and [14] for solving this.

Let $\Sigma \subset \mathbb{R}^3$ be a connected and properly embedded surface. Hence Σ is orientable and it separates $\mathbb{R}^3$ in two connected components. Assume Σ is oriented and let N be the unit normal vector field along Σ. Set $\Omega(N)$ the component of $\mathbb{R}^3 - \Sigma$ at which N points. So $\Sigma = \partial\Omega(N)$. We will apply the Alexandrov reflection method for the surface $S = \Sigma \cap \partial W$, where W is an open set in $\Omega(N)$.

Since $S \subset \Sigma$, it can be either non connected or non bounded. Therefore, we shall establish first what *a first (local) contact point* means.

Let $P \subset \mathbb{R}^3$ be a totally geodesic plane with unit normal vector field N_P. Let $p \in P$ be a point and $\gamma_{(p,N_P(p))}(t)$ the complete geodesic in $\mathbb{R}^3$ with initial conditions $\gamma_{(p,N_P(p))}(0) = p$ and $\gamma'_{(p,N_P(p))}(0) = N_P(p)$. If $\gamma_{(p,N_P(p))}(t)$ is disjoint from $\overline{W}$ for all t sufficiently large, let

$$\mathtt{p}_1(p) = \gamma_{(p,N_P(p))}(t_1) \tag{3.6}$$

be **the first contact point in** $\gamma_{(p,N_P(p))} \cap \overline{W}$, if this point exists, as t decreases from $+\infty$

Remark 7:
At arguments of reflection that involve the first contact point we are considering that t decreases, that is, the reflection planes went from "behind of the surface".

Let $p \in P$ be a point at which $\mathtt{p}_1(p)$ exists and is defined by (3.6). If $\mathtt{p}_1(p) \in S$, the intersection of $\gamma_{(p,N_P(p))}$ and S at $\mathtt{p}_1(p)$ can be either transversal or tangencial. If it is transverse and if $\gamma_{(p,N_P(p))}$ first leaves W through S, then we denote this point by

$$\mathtt{p}_2(p) = \gamma_{(p,v)}(t_2)\,. \tag{3.7}$$

If the intersection is tangential, let

$$\mathtt{p}_2(p) = \mathtt{p}_1(p)\,.$$

Definition 3.14:
Let $p \in P$ be such that there exist $\mathtt{p}_1(p)$ and $\mathtt{p}_2(p)$, i.e., $\gamma_{(p,N_P(p))}$ goes in and out of $\overline{W}$ through S. In this case, we say that p belongs to the **domain of the Alexandrov function**, *denoted by Λ, where the* **Alexandrov function associated to** *P is given by*

$$\alpha_1(p) = \frac{t_1 + t_2}{2}\,. \tag{3.8}$$

Geometrically, $t = \frac{t_1+t_2}{2}$ is the value for what the reflection of $\mathtt{p}_1(p)$, with respect to the plane $P(t)$, touches S at a first point.

Definition 3.15:
A ***first local reflection point of*** *S, with respect to the (oriented) plane P, is defined to be a*

point $\mathtt{p}_2(p)$ for which $p \in \Lambda \subset P$ is a local maximum of α_1. That is, there exists a neighborhood $U_p \subset \Lambda$ of p so that $\alpha_1(q) \leq \alpha_1(p)$ for all $q \in U_p$.

Definition 3.16:
*We say that α_1 has **an interior local maximum at** $p \in \Lambda$, if there exists a neighborhood $U_p \subset P$ of p such that, for all $q \in U_p$, one of the situations happens:*

- *$q \in \Lambda$ and $\alpha_1(q) \leq \alpha_1(p)$;*
- *$\gamma_{(q,N_P(q))} \cap \overline{W} = \emptyset$.*

*Any other local maximum of α_1 will be called a **boundary maximum** (for example, if $\gamma_{(p,N_P(p))}$ intersects ∂S).*

The next result justifies the previous definitions.

Lemma 3.17:
Let $\Sigma \in \mathcal{F}$ be a connected and properly embedded surface, and $P \subset \mathbb{R}^3$ be a plane with unit normal vector field N_P. If $p \in P$ is an interior local maximum for α_1 (with respect to the subsets $S \subset \Sigma$ and $W \subset \Omega(N)$), then the plane $P(\bar{t})$ is a plane of symmetry for Σ, here $\bar{t} = \alpha_1(p)$.

Proof. We will compare S to the reflection $\widetilde{S}^+(\bar{t})$ of S with respect to $P(\bar{t})$ (see Definition 3.6). By construction, $\mathtt{p}_2(p)$ is the reflection of $\mathtt{p}_1(p)$ with respect to $P(\bar{t})$.

Let $q \in \Lambda$ be a nearby point to p then, since p is a local maximum, we have

$$2\bar{t} - t_1(q) \geq t_2(q)\,.$$

This means that the reflection of $\mathtt{p}_1(q)$ with respect to $P(\bar{t})$, denoted by $\widetilde{\mathtt{p}}_1(q)$ is behind of $\mathtt{p}_2(q)$, and, since $q \in \Lambda$, we have $\widetilde{\mathtt{p}}_1(q) \in \overline{W}$. Hence, there exists a neighborhood of $\mathtt{p}_2(p)$ in $\widetilde{S}^+(\bar{t})$ contained in $\overline{W}$. In particular, if $\mathtt{p}_1(p) \neq \mathtt{p}_2(p)$, then S and $\widetilde{S}^+(t)$ have $\mathtt{p}_2(p)$ as an interior tangent point (see Figure 3.5). If $\mathtt{p}_1(p) = \mathtt{p}_2(p)$, then S and $\widetilde{S}^+(t)$ have $\mathtt{p}_1(p) = \mathtt{p}_2(p)$ as a boundary tangent point. Thus, from Theorem 3.4, we conclude the proof. □

We will see now that α_1 is a semicontinuos function with respect to planes as well as with respect to points.

Lemma 3.18:
Let S be closed and $\epsilon \to 0$ be a parameter. Suppose that $p_\epsilon \to p$ is a sequence of points contained in planes P_ϵ so that $P_\epsilon \to P$. Let α_1^ϵ be the corresponding sequence of Alexandrov functions. Then, if $\alpha_1^\epsilon(p_\epsilon)$ and $\alpha_1(p)$ exist, we have

$$\limsup_{\epsilon \to 0} \alpha_1^\epsilon(p_\epsilon) \leq \alpha_1(p)\,. \tag{3.9}$$

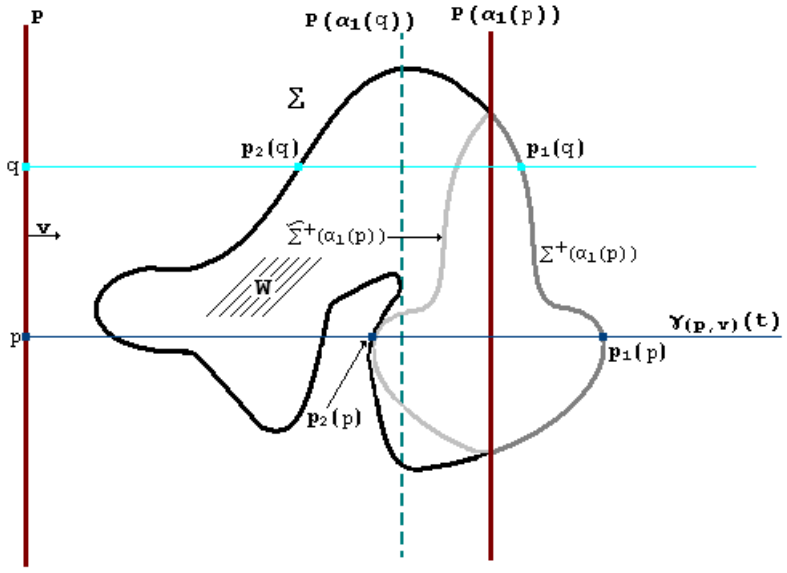


Figure 3.5: Local maximum por α_1

Proof. Let $(\mathtt{p}_1(p_\epsilon), \mathtt{p}_2(p_\epsilon))$ be the sequence of points in S associated to p_ϵ given by (3.6) and (3.7), respectively. Since S is closed, a subsequence of $(\mathtt{p}_1(p_\epsilon), \mathtt{p}_2(p_\epsilon))$ converges to a pair of points (Q_1, Q_2) in S (possibly $Q_1 = Q_2$), each above p. The heights, $(t_1^\epsilon, t_2^\epsilon)$ (of the points at the subsequence with respect to the planes P_ϵ) converge to the heights $(\bar{t}_1, \bar{t}_2)$ of the pair (Q_1, Q_2) with respect to P. Hence, from definition of $\mathtt{p}_1(p)$ and $\mathtt{p}_2(p)$, the heights (t_1, t_2) satisfy $t_i \geq \bar{t}_i$, for $i = 1, 2$. Consequently, (3.9) holds. □

Theorem 3.13 asserts that for any properly embedded annulus there exists a unique axial vector. In addition, this vector is the generator of the rulings of the cylinder. We already know how to extend the Alexandrov method for non-compact surfaces. Now, we will use this method for annular ends in properly embedded surfaces.

Set $\mathcal{F}$, where $\mathcal{F}$ is a family of surfaces in $\mathbb{R}^3$ satisfying the Hopf Maximum Principle. From the Cylindrically bounded Theorem (see Theorem 3.13), any properly embedded annulus $\Sigma \in \mathcal{F}$ is contained, up to an isometry, in the (solid) half-cylinder $C^+(R) = C^+(0, e_3, R)$. Denote by $E \subset \Sigma \in \mathcal{F}$ an annular end. Moreover, assume the boundary of E is contained in the xy-plane. Let $\Omega(N)$ be the component of E contained in $C(R)$.

We will prove that the first point of reflection contact for E occurs on the boundary of E. At a first sight, this affirmation could be surprising since the first point of reflection contact could occur *at infinity*. But, for proving the above assertion, we shall introduce a new object.

Definition 3.19:
Let P be a parallel plane to the axis e_3. Consider the height function $h(p) = \langle p, e_3 \rangle$ with respect to the xy-plane. We define the ***Alexandrov function associated to the end*** *E as*

$$\alpha(x) = max\{\alpha_1(p);\, h(p) = x\}\,, \tag{3.10}$$

where α_1 is the Alexandrov function associated to P given by (3.8)*.*

Now, we are in conditions to prove what we have announced.

Lemma 3.20:
Let $E \subset \Sigma \subset \mathbb{R}^3$ be an annular end of a properly embedded surface $\Sigma \in \mathcal{F}$ so that $E \subset C^+(R)$, where $\mathcal{F}$ is a family of surfaces in $\mathbb{R}^3$ satisfying the Hopf Maximum Principle. Let P be a parallel plane to the axis e_3. Then, one of the following facts holds:

- *$\alpha(x)$ is strictly decreasing;*
- *Σ has a plane of symmetry parallel to P.*

Proof. It is sufficient to prove that α is non increasing, since, in this case, α is either strictly decreasing or constant in some interval. In the later case, Lemma 3.17 guarantees the existence of a plane symmetry parallel to P.

To show α is non increasing, we will prove that $\alpha(x) \leq \alpha(0)$, for all $x > 0$, since the section $\{x = 0\}$ can be choose arbitrarily.

Claim 1: *$\alpha(x) \leq \alpha(0)$, for all $x > 0$, is equivalent to*

$$\left(\widetilde{E}(t) \cap \{x > 0\}\right) \subset \overline{W}\,, \tag{3.11}$$

for all $t > \alpha(0)$ (see Figure 3.6).

Proof of the Claim 1: If $\alpha(x) \leq \alpha(0)$ for all $x > 0$, the definition of the Alexandrov function (see Definition 3.19) implies that

$$\left(\widetilde{E}(t) \cap \{x > 0\}\right) \subset \overline{W}\,,$$

for all $t > \alpha(0)$.

Now, suppose, by contradiction, that there exist $x_0 > 0$ and $p \in P$, with $h(p) = x_0$ such that $\alpha_1(p) = \alpha(x) = t_0 > \alpha(0)$. If some neighborhood of $\widetilde{E}(t_0)$, containing $\widetilde{\mathbf{p}}_1(p)$, were contained in $\overline{W}$, the Maximum Principle would imply that $P(t_0)$ is a plane of symmetry for E, but it is impossible since $t > \alpha(0)$. Thus, Claim 1 is proved.

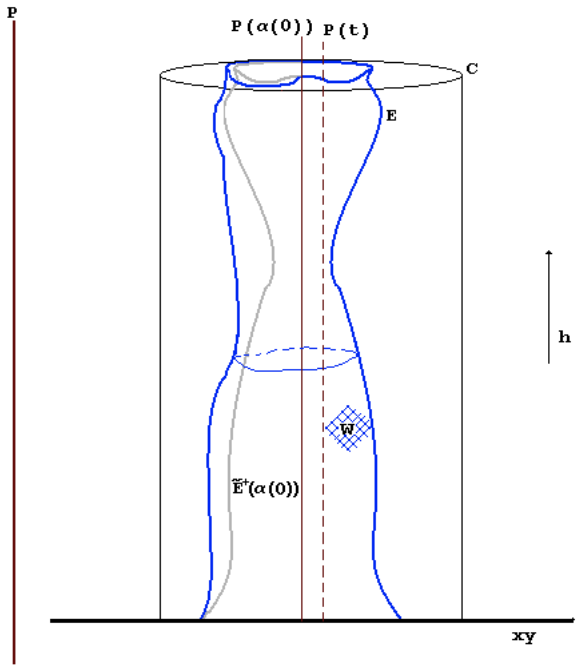


Figure 3.6: Claim 1

So, it is sufficiently to show (3.11). Let v be the normal vector along P and denote by v_ϵ the vector $v - \epsilon e_3$, for $\epsilon > 0$ sufficiently small. Let P_ϵ be the plane whose normal vector is v_ϵ, i.e., P_ϵ is tilted with respect to P, and it is clear that $P_\epsilon \to P$ as $\epsilon \to 0$.

If we reflect $E \cap P_\epsilon^+(t)$ through planes $P_\epsilon(t)$, the corresponding Alexandrov function α_1^ϵ should attain its maximum at the boundary, otherwise E would have a plane of symmetry parallel to P_ϵ , which is a contradiction. So, the functions α_1^ϵ does not have an interior maximum and $\alpha^\epsilon(x) \to -\infty$, as $x \to +\infty$.

Let z_ϵ be the maximum value that α_1^ϵ attains at the boundary. Then, we have

$$\left(\widetilde{E}_\epsilon(t) \cap \{x \geq 2R\epsilon\}\right) \subset \overline{W}\,, \tag{3.12}$$

for all $t \geq z_\epsilon$. The technical condition $x \geq 2R\epsilon$ implies that the projection of points in $\widetilde{E}_\epsilon(t) \cap P_\epsilon$ is in the domain of definition of α_1^ϵ (see Figure 7).

So, letting $\epsilon \to 0$, from (3.12), we get

$$\left(\widetilde{E}(t) \cap \{x > 0\}\right) \subset \overline{W}\,,$$

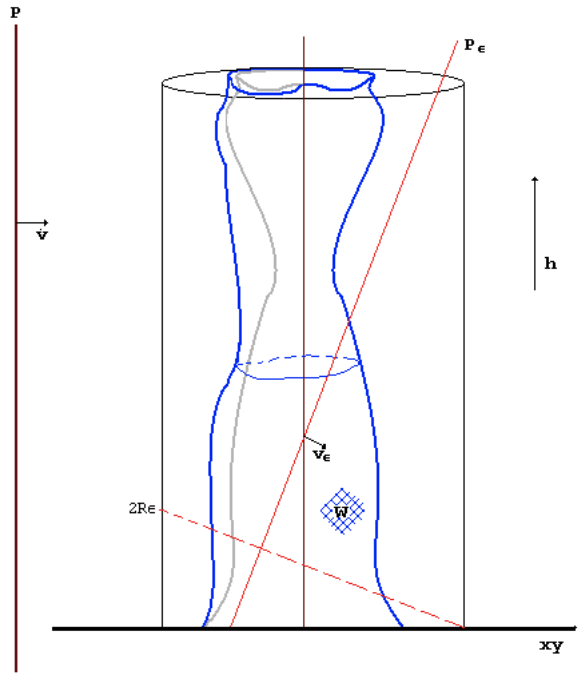


Figure 3.7: Tilted planes

for all $t \geq \limsup_{\epsilon\to 0} z_\epsilon$. But α_1 is upper semi-continuous, so

$$\limsup_{\epsilon\to 0} z_\epsilon \leq \alpha(0)\,,$$

and the Lemma is proved. □

3.7 Classification

We are able to prove the main result of this section:

Theorem 3.21:
Let $\mathcal{F}$ be a family of surfaces in $\mathbb{R}^3$ satisfying the Hopf Maximum Principle. If $\Sigma \in \mathcal{F}$ is a properly embedded surface with finite topology in $\mathbb{R}^3$, then every end of Σ is cylindrically bounded. Moreover, if $a_1, \ldots, a_k$ are the k axial vectors corresponding to the ends, then these vectors cannot be contained in an open hemisphere of $\mathbb{S}^2$. In particular,

- *$k = 1$ is impossible.*

- *If $k = 2$, then Σ is contained in a cylinder and is a rotational surface with respect to a line parallel to the axis of the cylinder (see Figure 3.8).*

- *If $k = 3$, then Σ is contained in a slab.*

Proof. If Σ is a properly embedded surface with finite topology, then its ends are annular type and the Cylindrically bounded theorem (Theorem 3.13) implies the first assertion.

Let $a_1, \dots a_k \in \mathbb{S}^2$ be the axial vectors corresponding to the ends. Suppose that all these vectors are contained in an open hemisphere, say $\mathbb{S}^2 \cap \{z > 0\}$. Let P be the xy-plane with normal vector $e_3 = (0, 0, 1)$. Consider $P(t)$ the foliation associated to P, as defined in Definition 3.6. As each axial vector points up, we can take $\bar{t} < 0$ sufficiently large such that $P(\bar{t}) \cap \Sigma = \emptyset$. Now, increasing $t > \bar{t}$, we will have a first contact point with the surface. Observe that this point is not at infinity because the axial vectors point up.

Now, do Alexandrov reflection with respect to the planes $P(t)$, let $t_1 > \bar{t}$ be the value of t where happens the first contact point. Then, for $t > t_1$, with t nearby to t_1, the reflection of the part $\Sigma \cap P(t)^-$ (that is, behind of $P(t)$) is a graph over $P(t)$. If we increase t, the part of Σ behind $P(t)$ is a graph as well. If this were not the case, we may argue as in Corollary 3.9, we will obtain the existence of an interior or (boundary) tangent point of the surface and its reflection. In both cases the Maximum Principle implies that the surface should be compact, which is a contradiction.

Therefore, as $t \to +\infty$, the part of Σ behind $P(t)$ is a graph over a domain of $P(t)$. But this contradicts the Height Estimates (Theorem 3.8), and the second assertion is proved.

Let us see the assertions about the ends.

- $k = 1$. It is clear, from the second assertion.

- $k = 2$. Since a_1 and a_2 can not be in an open hemisphere, we have $a_1 = -a_2$, and consequently Σ is contained in a cylinder with axis a_1. Up an isometry we can assume $a_1 = e_3$ and Σ is contained in $C = C(0, R, e_3)$, where O is the origin of $\mathbb{R}^3$. Let P be the xy-plane. Let $E_1 = \Sigma \cap P^+$ and $E_2 = \Sigma \cap P^-$ be the annular ends of the surface. Denote by Q a parallel plane to the axis of C, and let $\alpha^i, i = 1, 2$ the Alexandrov function associated to the end E_i. Apply Lemma 3.20 to each end, so we conclude that there exists a plane of symmetry of Σ parallel to Q, or α_1 attains a maximum local at a point $p \in \Sigma \cap \{z = 0\}$. In this last case, Lemma 3.17 guarantees that there exists a plane of symmetry of Σ parallel to Q. Thus, for any direction in $\mathbb{S}^1 \subset P$, one finds a plane of symmetry of Σ. So Σ is rotationally symmetric.

- $k = 3$. In this case the three axial vectors corresponding the ends should be contained in the same maximum circle of the sphere, otherwise, they would be contained in an open hemisphere, which is impossible. Consequently, it is clear, by Theorem 3.13, that the surface is contained in a slab determined by two parallel planes.

□

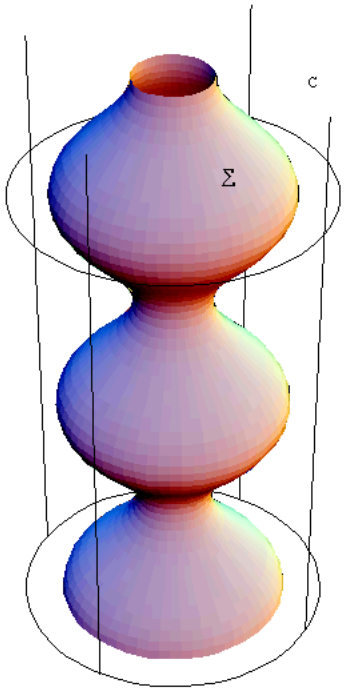


Figure 3.8: Delaunay surface.

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