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Formation of clusters and coarsening in locally attractive interacting particle systems

Gradient flows face-to-face – Granada

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joint work with Rishabh Gvalani, Greg Pavliotis and André Schlichting

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Hegselmann-Krause opinion dynamics

- For $i = 1, \dots, N$ consider

$$\mathrm{d}X_t^i = -\frac{\gamma}{N} \sum_{j=1}^{N} \nabla w\left(\frac{X_t^i - X_t^j}{\ell}\right) dt + \sqrt{2}\mathrm{d}B_t^i$$

on the periodic domain $S^1 \cong \mathbb{R}/\mathbb{Z} \cong [0,1]$, where

- the interaction potential $w : \mathbb{R} \to \mathbb{R}$ is given by

$$w(x) = \begin{cases} \frac{1}{2}\left(x^2 - 1\right) & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| > 1, \end{cases}$$

- $\ell > 0$ is the interaction range and $\gamma > 0$ the interaction strength
- $B_t^1, \dots, B_t^N$ are independent standard Brownian motions in $\mathbb{R}$.

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  - $\ell > 0$ is the interaction range and $\gamma > 0$ the interaction strength
  - $B_t^1, \dots, B_t^N$ are independent standard Brownian motions in $\mathbb{R}$.
- Corresponding mean-field PDE:

$$\partial_t \rho := \Delta\rho + \nabla \cdot \left((\nabla W_{\gamma,\ell} * \rho)\rho\right) \qquad \text{on } [0,1] \qquad \text{where } W_{\gamma,\ell}(x) = \gamma\ell w\left(\frac{x}{\ell}\right)$$

- This PDE is a W_2-gradient flow for the free energy

$$\mathcal{F}(\rho) = \int_{[0,1]} \rho \log \rho + \frac{1}{2} \iint_{[0,1]^2} W_{\gamma,\ell}(x-y)\rho(x)\rho(y)\mathrm{d}x\mathrm{d}y\,.$$

- Free energy dissipation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(\rho_t) = -\int_{[0,1]} \left|\nabla(\log \rho_t + W_{\gamma,\ell} * \rho_t)\right|^2 \rho_t \mathrm{d}x \leq 0.$$

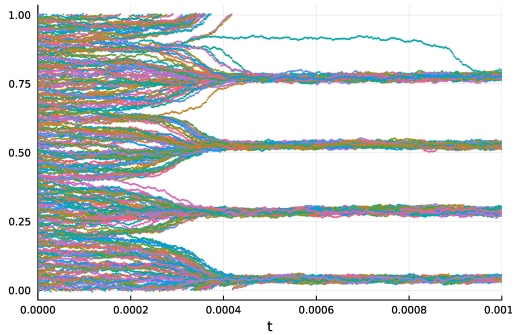
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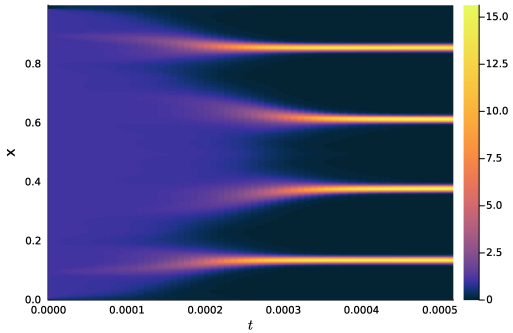
Effect I: Initial clustering

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Microscopic model



Mean-field PDE



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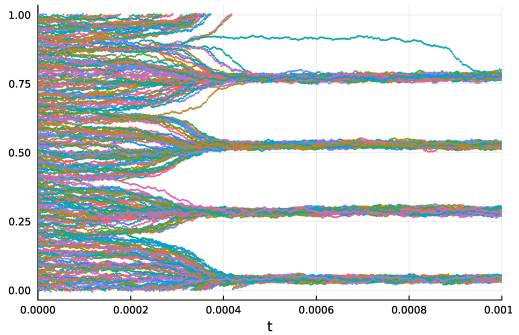
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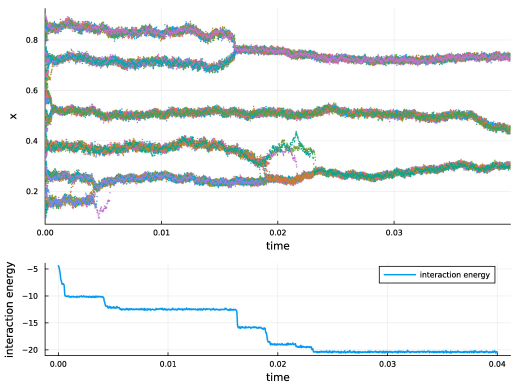
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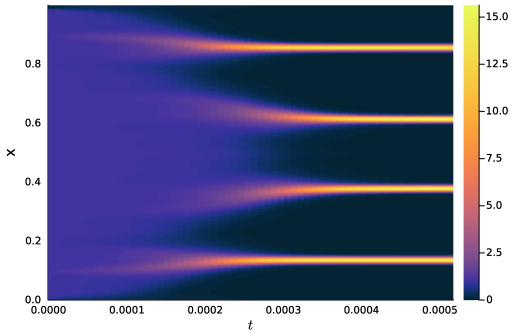
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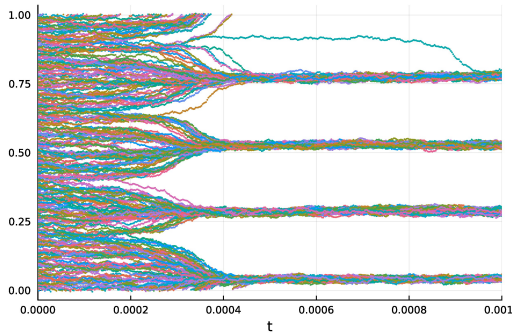


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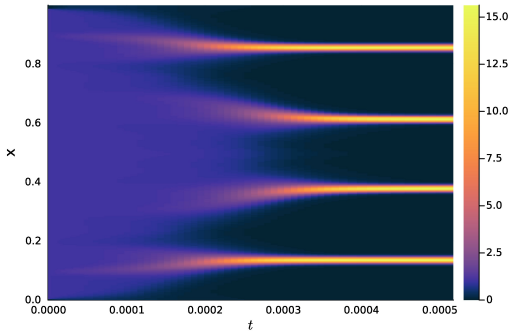


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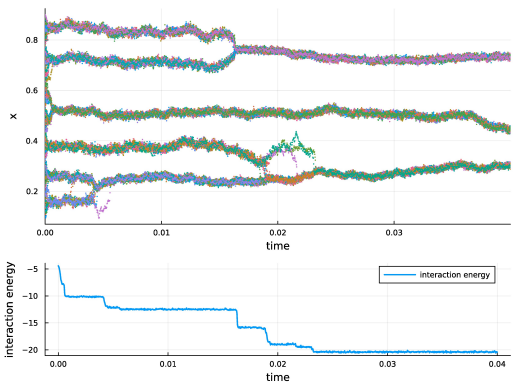


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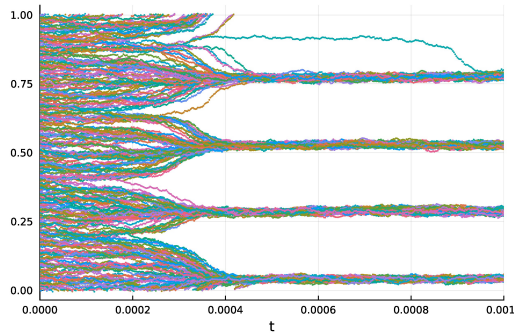
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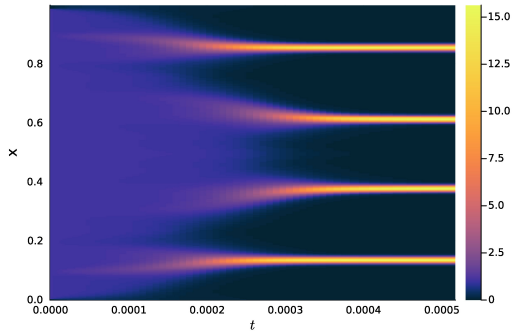
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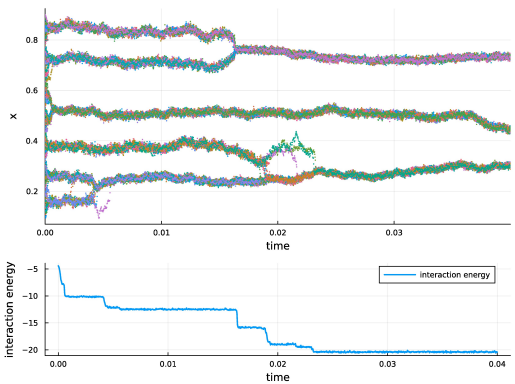


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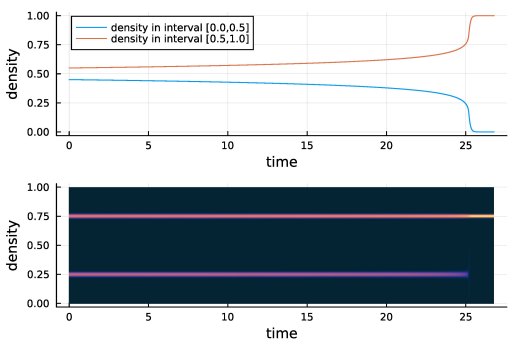


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Effect III: Mass exchange

- Timescale $e^{\gamma \ell m/2}$ for clusters of mass $m \in (0, 1)$
- Dynamical metastability (Otto and Reznikoff 2007)

also present, but needs a lot of particles to be visible



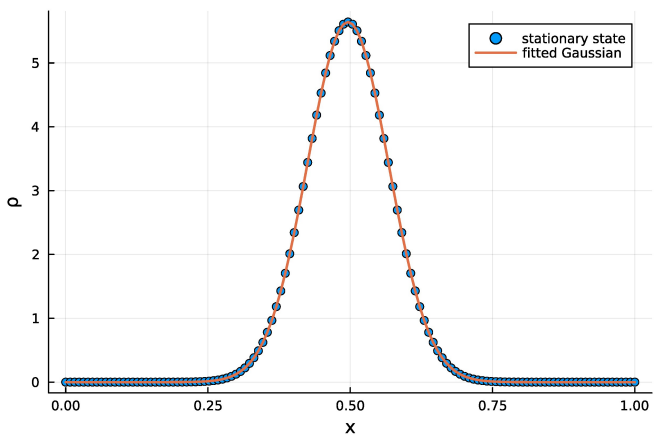
Focus of this talk

Proof: All global minimizers of the free energy are symmetrically decreasing

Free energy:

$$\mathcal{F}(\rho) = \int_{[0,1]} \rho \log \rho + \frac{1}{2} \iint_{[0,1]^2} W_{\gamma,\ell}(x-y)\rho(x)\rho(y)\mathrm{d}x\mathrm{d}y\,.$$

Numerical steady state: Gaussian of variance $\sigma^2 = \frac{\ell}{\gamma w''(0)}$.



Non-trivial global minimizers of the free energy are symmetrically decreasing

Theorem

Assume that[1] $w(x) := \frac{1}{2}(x^2 - 1)\chi_{[-1,1]}(x)$.
Let ρ *be a global minimizer of the free energy.*
Then there exists a rotation $T : S^1 \to S^1$ *such that* $\rho = \rho^* \circ T$, *where* ρ^* *is the symmetric decreasing rearrangement of* ρ.

[1] or more generally that $w : \mathbb{R} \to \mathbb{R}$ continuous, even, symmetrically non-decreasing, w has compact support, w is C^2 within its support and attains its global minimum at 0 with $w''(0) > 0$.

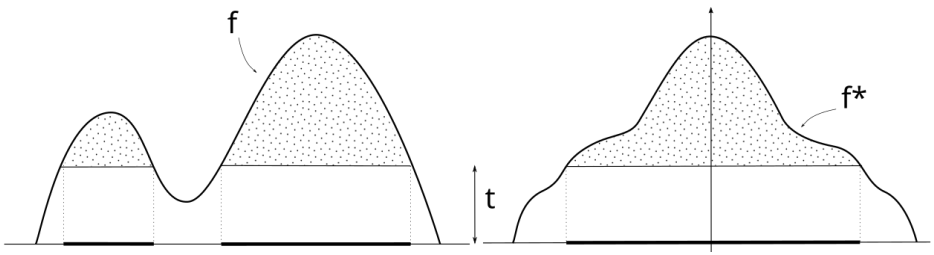
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Symmetric decreasing rearrangement of $\rho : S^1 \to [0, \infty)$:

- Let $\overrightarrow{n} = (1, 0) \in S^1$ and $d : S^1 \times S^1 \to [0, \infty)$ be the geodesic distance on S^1.



- The symmetric decreasing rearrangement ρ^* is the unique measurable $\rho^* : S^1 \to [0, \infty)$
  - which is a non-increasing function of $d(\bullet, \overrightarrow{n})$
  - and equimeasurable with ρ, that is, level sets of ρ^* have the same masses as the those of ρ.

[1] or more generally that $w : \mathbb{R} \to \mathbb{R}$ continuous, even, symmetrically non-decreasing, w has compact support, w is C^2 within its support and attains its global minimum at 0 with $w''(0) > 0$.

Proof

Free energy: $\mathcal{F}(\rho) = S(\rho) + I(\rho)$ where

- entropy: $S(\rho) := \int_{[0,1]} \rho(x) \log(\rho(x)) \mathrm{d}x$
- interaction energy: $I(\rho) := \frac{1}{2} \iint_{[0,1]^2} \rho(x)\rho(y) W_{\gamma,\ell}(x-y) \mathrm{d}x\mathrm{d}y$

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Entropy depends only on masses of level sets $\Rightarrow S(\rho) = S(\rho^*)$

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Theorem (Riesz rearrangement inequality[2])

Let $K : [0,\pi] \to [0,\infty)$ be a non-decreasing function. Then, for all measurable non-negative functions f, g on S^1 it holds that

$$\int_{S^1 \times S^1} f(x)g(y)K(d(x,y))\mathrm{d}x\mathrm{d}y \geq \int_{S^1 \times S^1} f^*(x)g^*(y)K(d(x,y))\mathrm{d}x\mathrm{d}y.$$

This gives $\mathcal{F}(\rho) \overset{\text{Riesz}}{\geq} \mathcal{F}(\rho^*) \overset{\rho \text{ minimizer}}{\geq} \mathcal{F}(\rho).$ $\Rightarrow$ $\mathcal{F}(\rho) = \mathcal{F}(\rho^*).$

How can we show that ρ equals ρ^* up to rotation?

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Variant of the strict Riesz rearrangement inequality

- Equality case of Riesz rearrangement inequality for strictly increasing kernels K was handled in Beckner 1993[3], Burchard 1996[4]

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- Interaction potentials $W_{\gamma,\ell}(x-y) \widehat{=} K(d(x,y))$ considered here are not strictly increasing because of compact support

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Lemma (Variant of strict Riesz rearrangement inequality)

Let $K : [0,\pi] \to [0,\infty)$ be increasing.
Let K be strictly increasing on $[0,\delta]$ for some $\delta > 0$.
Let $\rho : S^1 \to [0,\infty)$ be an analytic function for which

$$\int_{S^1 \times S^1} \rho(x)\rho(y)K(d(x,y))\mathrm{d}x\mathrm{d}y = \int_{S^1 \times S^1} \rho^*(x)\rho^*(y)K(d(x,y))\mathrm{d}x\mathrm{d}y.$$

Then there exists a rotation $T : S^1 \to S^1$ such that $\rho = \rho^* \circ T$.

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Proof Part I: Polarizations

Main idea: Analyticity allows to 'globalize' local strictness

Variant of Beckner's polarization argument[5]:

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- Let $M^0 = S^n \cap H$. Split S^1 into two open halfspheres M^+ and M^- such that $S^1 = M^+ \dot{\cup} M^0 \dot{\cup} M^-$ and $\overrightarrow{n} \in M^+$ where $M^0 = S^1 \cap H$.
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- Let $\sigma : S^1 \to S^1$ be the reflection with respect to H
- We define the polarization $\rho^\sigma : S^n \to [0, \infty)$ by

$$\rho^\sigma(x) := \begin{cases} \max\{\rho(x), \rho(\sigma x)\} & \text{if } x \in M^+ \\ \rho(x) & \text{if } x \in M^0 \\ \min\{\rho(x), \rho(\sigma x)\} & \text{if } x \in M^-. \end{cases}$$

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Proof Part II: Conclusion

Using elementary manipulations, it is possible to show (Beckner 1993; Burchard 2009)

$$
\begin{aligned}
& I(\rho) - I(\rho^{\sigma}) \\
= & \int_{M^+ \times M^+} \underbrace{\max\Big\{\big(\rho(\sigma x) - \rho(x)\big)\big(\rho(y) - \rho(\sigma y)\big), 0\Big\}\Big\{K(d(x,\sigma y)) - K(d(x,y))\Big\}}_{\geq 0} \mathrm{d}x\mathrm{d}y.
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Since K is increasing and strictly increasing on $[0, \delta]$:

$$\big(\rho(x) - \rho(\sigma x)\big)\big(\rho(y) - \rho(\sigma y)\big) \geq 0 \quad \text{for all } x, y \in M^+ \quad \text{with } d(x,y) < \delta.$$

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$$\Rightarrow \quad \rho \geq \rho \circ \sigma \quad \text{or} \quad \rho \leq \rho \circ \sigma \quad \text{on } M^+.$$
$$\Rightarrow \quad \rho = \rho^\sigma \quad \text{or} \quad \rho = \rho^\sigma \circ \sigma \quad \text{on } S^1.$$

Conclude using:

Lemma

If $\rho = \rho^\sigma$ or $\rho = \rho^\sigma \circ \sigma$ for all reflections σ, then there exists a rotation $T : S^n \to S^n$ such that $\rho = \rho^* \circ T$. *(Burchard 2009, Lemma 2.11)*

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Thank you for your attention!

References I

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