The Heat Equation in the Hyperbolic Space. Asymptotic Behaviour.

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Gradient Flows - Face to Face Workshop

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The Equation

Introduction

The Equation

The goal of this presentation is to showcase some recent results regarding the behaviour of the solutions of the heat equation posed in the hyperbolic space $\mathbb{H}^d$,

$$\begin{cases} u_t = \Delta_{\mathbb{H}^d} u, & x \in \mathbb{H}^d, t > 0, \\ u(0,\cdot) = u_0 \geq 0, & x \in \mathbb{H}^d, \end{cases} \tag{1}$$

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The Hyperbolic Space

The Riemannian manifold $\mathbb{H}^d$ is the unique simply connected, d-dimensional Riemannian manifold of constant sectional curvature equal to -1. As such, it admits several different coordinate systems $\vec{x} = (x_1, ..., x_d) \in \mathbb{H}^d$.

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Given one set of coordinates, the tensor of the Riemannian metric $\mathrm{d}s^2$ will be denoted as g, its entries will be denoted as $g_{i,j}$, and the entries of the inverse matrix as $g^{i,j}$. The Laplace-Beltrami operator then reads

$$\Delta_{\mathbb{H}^d} u := \frac{1}{\sqrt{\det(g)}} \sum_{i=1}^{d} \left(\frac{\partial \left(\sqrt{\det(g)} \sum_{k=1}^{d} \left(g^{i,k} \frac{\partial u}{\partial x_k} \right) \right)}{\partial x_i} \right).$$

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$$\mathrm{d}\mu(\vec{x}) = \sqrt{\det(g(\vec{x}))}\ \mathrm{d}\vec{x} \quad \text{and so, } u_0 \in L^1(\mathbb{H}^d) \Leftrightarrow \int_{\mathbb{H}^d} |u_0|\, \mathrm{d}\mu(\vec{x}) < \infty$$

The Hyperbolic Space. Polar (or elliptic) coordinates

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$$u_t = \partial^2_{rr} u + (d-1)\coth(r)\partial_r u$$

and $u(t,r,\theta) = u(t,r)$. We will refer to this case as radial solutions .

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In the cases where the problem can be reduced to a one-dimensional case, existence and regularity are straightforward.

For a more general initial datum $u_0 \in L^1(\mathbb{H}^d)$ a unique solution u of problem (1) exists, and $u \in C([0,\infty) : L^1(\mathbb{H}^d))$ and $u \in C^\infty((0,\infty) \times H^d)$.

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In particular, a kernel $P_d^y(t,x)$ exists, in the sense that it is the fundamental solution of (1) with initial datum $u_0(x) = \delta_y(x)$, and the solution u for another integrable datum is given by

$$u(t,x) = \int_{\mathbb{S}^{d-1}} \int_0^\infty u_0(y) P_d^y(t,x) \, \mathrm{d}\mu(r,\theta).$$

Existence, uniqueness, kernel

Moreover, given the polar coordinates $r \geq 0$, $\theta \in \mathbb{S}^{d-1}$, the explicit formula for the kernel is known,

$$P_d^y(t,x) = P_d(t, \lambda(x,y)), \quad \text{where} \quad \lambda(x,y) = \text{dist}(x,y)$$

and P_d and λ both have closed formulas.

See to this respect Strichartz (1983), Davies & Mandouvalos (1988), the notes by Grygor'yan & Noguchi (1997$\sim$ 1998) and the book by Grygor'yan (2009).

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and, as a consequence of the $L^1 - L^\infty$ smoothing effect,

$$\|u(t,x) - \mathcal{M}P_d(t,x)\|_{L^\infty(\mathbb{H}^d)} = O(t^{-\frac{3}{2}} e^{-\lambda_1}),$$

where λ_1 represents the first eigenvalue of the Laplace-Beltrami operator in $\mathbb{H}^d$ (in contrast to the euclidean case). Here,

$$\lambda_1 = \frac{(d-1)^2}{4}. \tag{2}$$

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$$t^{\frac{3}{2}} e^{\lambda_1 t} |u(t,x) - \mathcal{M} P_d^y(t,x)| = Q^y(x) > 0 \quad \text{for all } x \in \mathbb{H}^d, \text{ for any } y \in \mathbb{H}^d,$$

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and consider an initial datum $u_0(x) = \delta_0(x) + \delta_y(x)$.

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2. Can we obtain a speed of convergence in the L^1 norm? That would translate directly into an improved convergence speed in the L^∞ norm through the $L^1 - L^\infty$ smoothing effect.
3. Most importantly: What is the large-times behaviour of integrable, non-radial solutions?

Entropy techniques

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The equation satisfied by v can then be written as

$$\begin{cases} v_\tau = \partial_\rho \left(v \cdot \partial_\rho \left[\ln \left(\frac{v}{\mathcal{V}} \right) \right] \right), & \rho \geq \rho_0(\tau),\ \tau > 0, \\ v(\tau, \rho_0(\tau)) = 0, & \tau > 0, \\ v(0, \cdot) = v_0, & \rho \geq \rho_0(\tau). \end{cases}$$

where $\rho_0(t) = -(d-1)e^{\frac{\tau}{2}}$.

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Next we define the Entropy of v relative to the "equilibrium" $\mathcal{V}$,

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and then we prove (after **a lot** of computations) that

$$\partial_\tau H(v|\mathcal{V}) \leq k_d \ \mathcal{M}_{\mathcal{R}} \ e^{\tau - m_d e^\tau} - H(v|\mathcal{V}),$$

where k_d, m_d are constants that depend on the dimension, and we defined $\mathcal{M}_{\mathcal{R}} := \int_0^\infty u_0(r) \sinh^{d-1}(r) \ \mathrm{d}r$.

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Decay of $H(v|\mathcal{V}) \to$ Decay of $\|v - \mathcal{V}\|_{L^1} \to$ Decay of $\|u - \mathcal{U}\|_{L^1(\mathbb{H}^d)}$

for a certain profile $\mathcal{U}(t, x)$ that will be specified later.

Results for radial solutions

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Convergence for radial solutions

(Cañizo, G., Marín) Convergence for radial solutions.

Let u_0 be radial and $\mathcal{M} < \infty$. Then there exist a value $C_\infty > 0$ and a t_0 such that, for all $t \geq t_0$,

$$\|u(t,r) - C_\infty\, \Gamma(t, r + (d-1)t)\|_{L^1(\mathbb{H}^d)} \lesssim t^{-\frac{1}{2}},$$

where $\Gamma(t,x)$ is the 1-d gaussian function and $C_\infty = \frac{2^{d-2}}{\sqrt{\pi}} \mathcal{M}_{\mathcal{R}}$.

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Now, the next logical question is what happens in the more general case $u_0 \in L^1(\mathbb{H}^d)$.

Results for integrable solutions

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First steps

As commented at the beginning of the presentation, let us consider polar coordinates $x = x(r_x, \theta_x)$, with $r_x > 0$, $\theta_x \in \mathbb{S}^{d-1}$. Also, we fix the origin as the "privileged pole" and define $P_d(t, x)$ the kernel in polar coordinates around the origin.

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Since we know that, due to the ballistic motion, the mass of the solution escapes really fast towards infinity "in the direction of r", one can guess there is very little time for the solution to spread "in the direction of θ". As a consequence, the solution must have a memory of the initial distribution of the mass of u_0 depending on θ, call it $\Phi(\theta_x)$.

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Now the question is, what should this φ be?

Computing φ

Since $u(t,x) = \int u_0 P_d^y \, \mathrm{d}\mu(y)$, we have

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L^1 convergence

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Remember, the mass is located at distance $r\sim(d-1)e^\tau$.

Improving the L^1 convergence

This time we study the Entropy of v relative to the product $\mathcal{N}(\tau,\theta)\mathcal{V}(\tau,r)$,

$$H(v|\mathcal{N}\mathcal{V}) := \int_{\mathbb{S}^{d-1}} \int_{\rho_0(t)}^{\infty} v \cdot \ln\left(\frac{v}{\mathcal{N}\mathcal{V}}\right) \ \mathrm{d}\rho \ \mathrm{d}\theta,$$

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But there is a caveat: due to the structure of the quantity $\partial_\tau H$, we are in need of a very fast convergence of $\mathcal{N}(\tau,\theta)$ to a certain limit in time.

Convergence of $\mathcal{N}$ and H

After many computations, and with an extra hypothesis on the initial datum (an extra momentum), we can prove that there exists a $\mathcal{N}_\infty(\theta) \in L^\infty(\mathbb{S}^{d-1})$ such that

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Again, we prove that there exists a value $H^*(v_0)$, such that

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$$\|v - \mathcal{NV}\|_{L^1} \leq \sqrt{H^*(v_0)}e^{-\frac{\tau}{2}} \Rightarrow \|u - C_\infty \mathcal{N}_\infty \Gamma\|_{L^1} \leq \sqrt{H^*(v_0)}t^{-\frac{1}{2}}$$

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There must be, certainly, a relationship between Φ and $\mathcal{N}_\infty$.

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Indeed, we are able conclude that $\Phi(\theta) = |\mathbb{S}^{d-1}| \cdot \mathcal{N}_\infty(\theta)$ a.e.,

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Our last result follows.

L^1 convergence with speed

(Cañizo, G., Marín) L^1 convergence with rates.

Let $u_0 \in L^1(\mathbb{H}^d)$ with some momentum and $\Phi \in L^1(\mathbb{S}^{d-1})$. Then,

$$\|u(t,x) - C(t)\,\mathcal{N}(t,\theta)\,\Gamma(t, r+(d-1)t)\|_{L^1(\mathbb{H}^d)} \lesssim t^{-\frac{1}{2}}.$$

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Moreover, there exists a time t_0 such that, for all $t > t_0$,

$$\|u(t,x) - \underbrace{\frac{2^{d-2}}{\sqrt{\pi}}}_{=C_\infty} \underbrace{|\mathbb{S}^{d-1}|^{-1}\,\Phi(\theta)}_{=\mathcal{N}_\infty(\theta)}\,\Gamma(t, r+(d-1)t)\|_{L^1(\mathbb{H}^d)} \lesssim t^{-\frac{1}{2}}.$$

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If we want to express convergence in terms of the heat kernel G,

$$\|u(t,x) - \Phi(\theta)\, P_d(t,r)\|_{L^1(\mathbb{H}^d)} \lesssim t^{-\frac{1}{2}} \quad \text{for all} \quad t \geq t_0.$$

The End

Thank you very much for your attention.

The Hyperbolic Space. Hyperbolic coordinates

Let $\mathcal{P} \subset \mathbb{H}^d$ a hyperplane isomorphic to $\mathbb{H}^{d-1}$, and $\vec{x} = (r, w)$, where $r = \text{dist}(x, \mathcal{P}) \in (-\infty, \infty)$ and $w \in \mathbb{H}^{d-1}$.

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The Hyperbolic Space. Hyperbolic coordinates

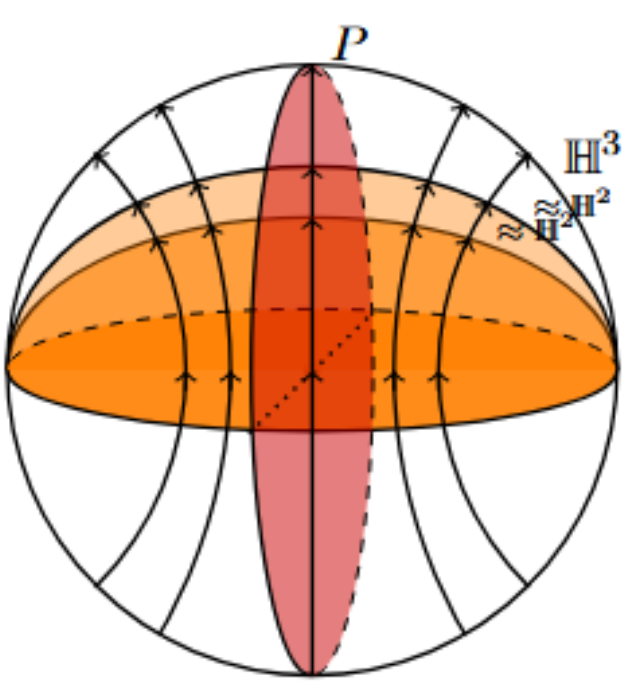
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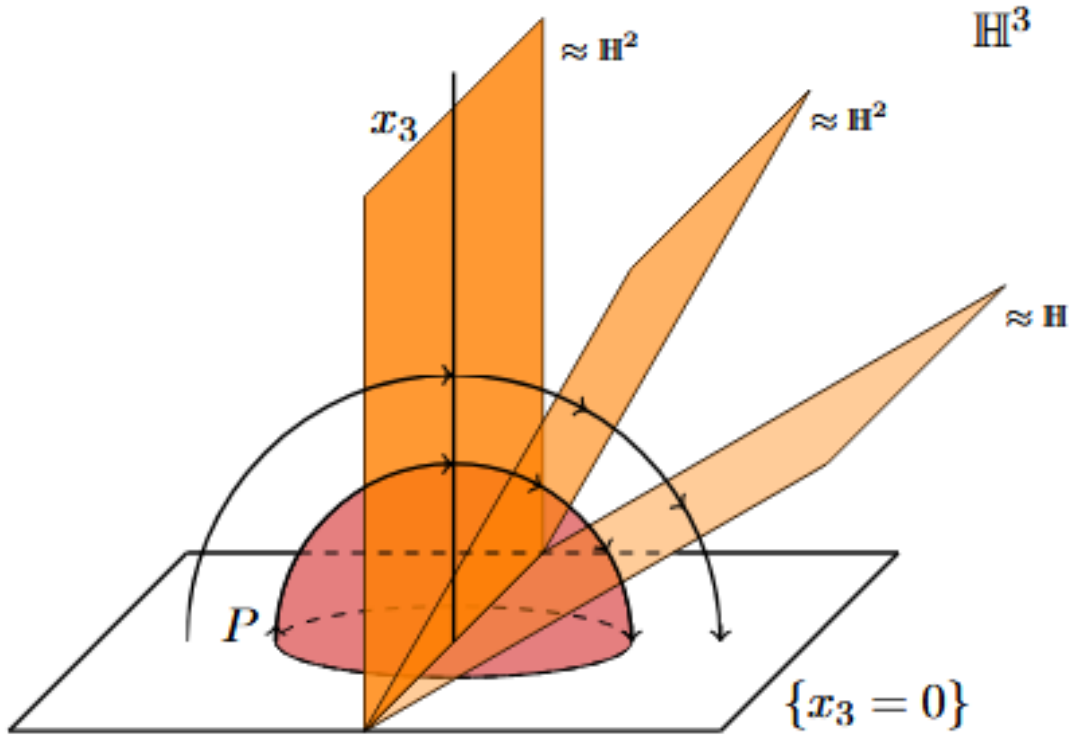
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$$\mathrm{d}\mu(r, w) = \cosh^{d-1}(r) \ \mathrm{d}r \, \mathrm{d}w$$

The Hyperbolic Space. Hyperbolic coordinates



(a) Poincaré disk model.



(b) Poincaré half-space model.

Figura: Arrowed curves represent geodesics parametrized by $r \in (-\infty, \infty)$. In orange some *half-rugby balls*, a popular name for the subspaces $\mathbb{H}^{d-1}$. In red, the hyperplane $\mathcal{P}$. Image by I. Gonzálvez

The Hyperbolic Space. Hyperbolic coordinates

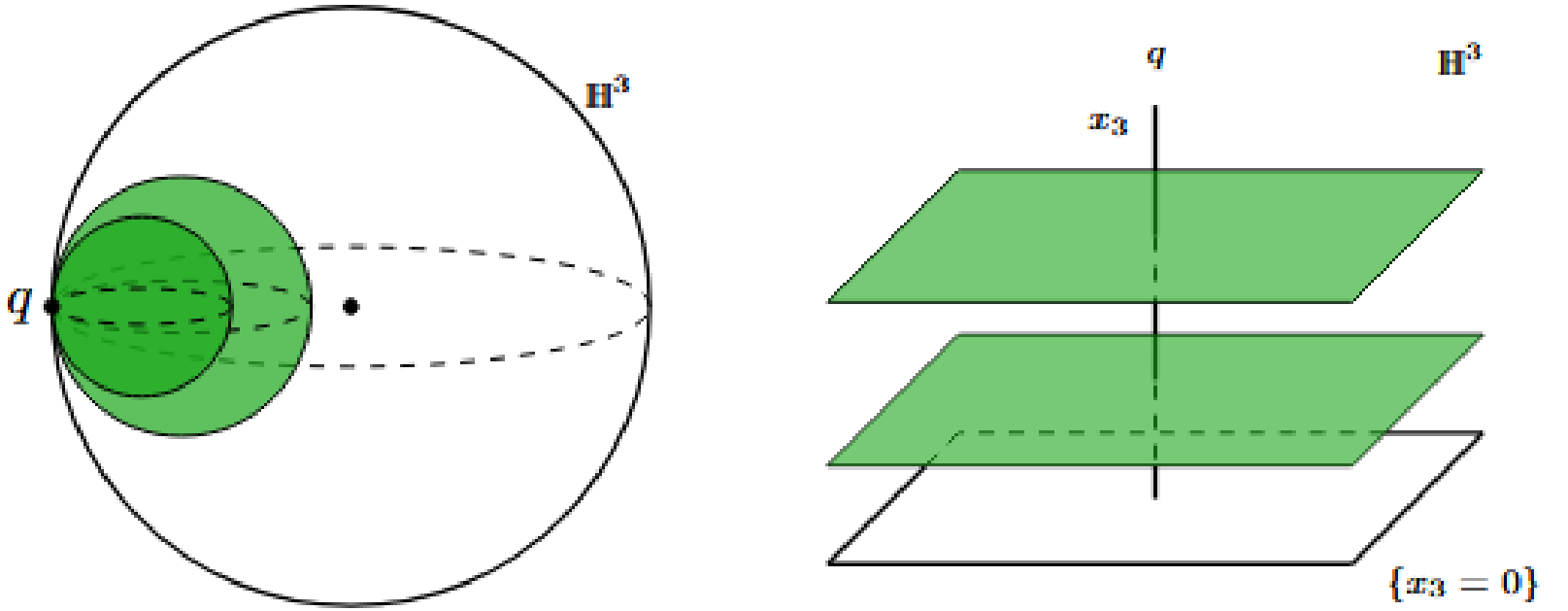
As before, in the case where $u_0(r, w) = u_0(r)$, then the problem is one-dimensional and the equation is

$$u_t = \partial^2_{rr} u + (d-1)\tanh(r)\partial_r u$$

and $u(t, r, w) = u(t, r)$. We will refer to this case as planar solutions .

The Hyperbolic Space. Parabolic coordinates

In the hyperbolic space there exists a certain type of objects that do not belong in other spaces like $\mathbb{R}^d$. We are talking about the horospheres.



(A) Horospheres centred at q in the disc model

(B) Horospheres centred at q in the upper-plane model

Figura: Images by I. Gonzálvez

The Hyperbolic Space. Parabolic coordinates

Let $\mathcal{H} \subset \mathbb{H}^d$ a horosphere isomorphic to $\mathbb{R}^{d-1}$, and $\vec{x} = (r, w)$, where $r = \text{dist}(x, \mathcal{H}) \in (-\infty, \infty)$ and $w \in \mathcal{H}$.

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Let $g = \begin{pmatrix} 1 & 0 \\ 0 & e^{2r} \cdot I_{d-1} \end{pmatrix}$. Then,

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Let $g = \begin{pmatrix} 1 & 0 \\ 0 & e^{2r} \cdot I_{d-1} \end{pmatrix}$. Then,

$$\Delta_{\mathbb{H}^d} u = \frac{1}{e^{(d-1)r}} \partial_r \left(e^{(d-1)r} \cdot \partial_r u \right) + \frac{1}{e^{2r}} \Delta_{\mathbb{R}^{d-1}} u,$$

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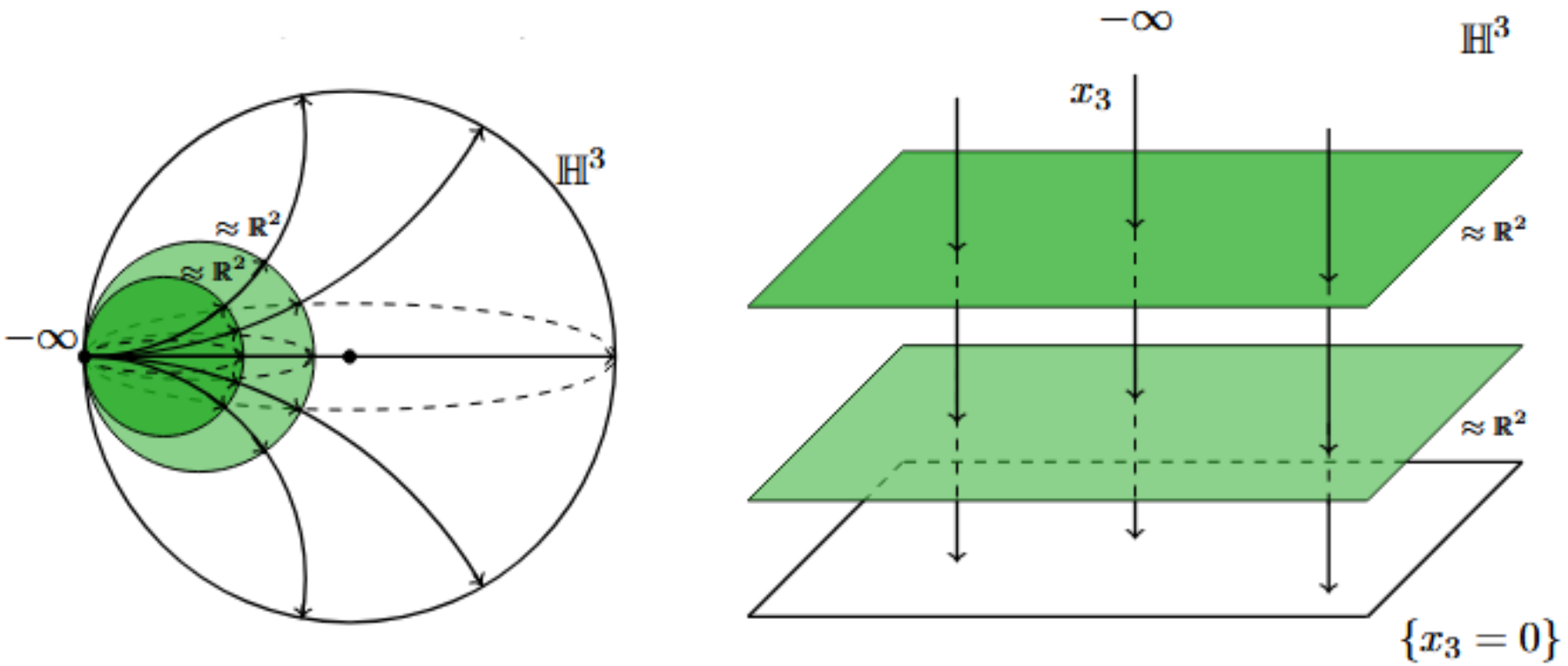
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$$\mathrm{d}\mu(r, w) = e^{(d-1)r} \ \mathrm{d}r \, \mathrm{d}w$$

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(a) Poincaré disk model.

(b) Poincaré half-space model.

Figura: Arrowed curves represent geodesics parametrized by $r \in (-\infty, \infty)$. In green some horospheres. Image by I. Gonzálvez

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And again, in the case where $u_0(r, w) = u_0(r)$,

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These ones do not have a parallel in the Euclidean Space!!!

The initial data

Initial datum	**Coord.**	**Metric** $\mathrm{d}\mu$
u_0 radial	Polar	$\sinh^{d-1}(r)\ \mathrm{d}r$
u_0 planar	Hyperbolic	$\cosh^{d-1}(r)\ \mathrm{d}r$
u_0 horospheric	Parabolic	$e^{(d-1)r}\ \mathrm{d}r$
$u_0 \in L^1(\mathbb{H}^d)$	Polar	$\sinh^{d-1}(r)\ \mathrm{d}r\,\mathrm{d}\theta$

Planar and horospheric solutions

(Cañizo, G., Marín) L^1 convergence for planar solutions.

Let u_0 be planar and $\mathcal{M}_{\mathcal{P}} < \infty$. Then there exist a value $C_\infty > 0$ and a t_0 such that, for all $t \geq t_0$,

$$\|u(t,r) - C_\infty\, \Gamma(t, r + (d-1)t)\|_{L^1(\mathbb{R};\, \mathrm{d}\mu(r))} \lesssim t^{-\frac{1}{2}}.$$

(Cañizo, G., Marín) L^1 convergence for horospheric solutions.

Let u_0 be horospheric and $\mathcal{M}_{\mathcal{H}} < \infty$. Then there exist a value $C_\infty > 0$ and a t_0 such that, for all $t \geq t_0$,

$$\|u(t,r) - C_\infty\, \Gamma(t, r + (d-1)t)\|_{L^1(\mathbb{R};\, \mathrm{d}\mu(r))} \lesssim t^{-\frac{1}{2}}.$$

The REAL End

Thank you very much, again, for your attention.