A Li-Yau and Aronson-Bénilan approach to study Keller-Segel

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Joint work with C. Elbar & F. Santambrogio

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Keller-Segel

$$\begin{cases} \partial_t \rho = \Delta \rho^m - \operatorname{div}(\rho \nabla u) & \text{in } (0,\infty) \times \mathbb{R}^d, \\ -\Delta u = \rho & \text{in } (0,\infty) \times \mathbb{R}^d, \end{cases} \tag{KS}$$

where $m = 2 - \frac{2}{d}$.

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Goals:

➢ Dimension $d = 2$. A Li-Yau type estimate.

➢ **Li-Yau** [LY86]. The heat equation $\partial_t \rho = \Delta \rho = \operatorname{div}(\rho \nabla(\log \rho))$,

$$\Delta(\log \rho(t,\cdot)) \geq -\frac{d}{2t}.$$

✍ Li-Yau for **non-local aggregation**-diffusion???

➠ L^∞ bounds on ρ in terms of $\delta := \inf_{x \in \mathbb{R}^d} \Delta v(x)$.

Overview

1. Differential inequality on the pressure.
2. L^∞ estimates.
3. Li-Yau estimates.
4. Discussions and new implications.

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$$v(t,x) = \log \rho(t,x) - u(t,x).$$

Let $x_0 = x_0(t)$ be a point achieving the minimum of $\Delta v(t,\cdot)$

☞ **Differential inequality**.

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta v(t,x_0) \geq (\Delta v(t,x_0))^2 - \frac{(d-1)^2}{2d}\sup_{i,j}\|\partial_{ii}u - \partial_{jj}u\|_{L^\infty}^2$$

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☞ **Estimate** $\|\partial_{ii}u - \partial_{jj}u\|_{L^\infty}$.

$$\sup_{i,j}\|\partial_{ii}u - \partial_{jj}u\|_{L^\infty} \lesssim |\widetilde{\delta}|^{d/(d+2)}\|\rho\|_{L^2}^{4/(d+2)},$$

where $\delta = \min(\Delta v)$ and $\widetilde{\delta} = \max(\delta, \|\rho\|_{L^\infty})$

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L^∞ estimates

Let us remember that

$$v(x) = \log \rho(x) - u(x) \quad \text{and} \quad \Delta v(x) = \Delta \log \rho(x) + \rho(x).$$

We want to study L^∞ estimates in terms of

$$\delta := \inf_{x \in \mathbb{R}^d} \Delta v(x).$$

- **Small mass**. Assume $0 < M < \frac{8\pi}{e}$.
- **Subcritical mass**. Assume $0 < M < 8\pi$.
- **Critical mass**. Assume $M = 8\pi$.

Small mass

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☞ Arbitrary x_0 and $r > 0$

$$\log \rho(x_0) \leq \fint_{B_r(x_0)} \log \rho + \frac{|\widetilde{\delta}|}{8} r^2 \qquad \text{[Subharmonic]}$$

$$\leq \log \left(\fint_{B_r(x_0)} \rho \right) + \frac{|\widetilde{\delta}|}{8} r^2 \leq \log \left(\frac{M}{\pi r^2} \right) + \frac{|\widetilde{\delta}|}{8} r^2 \qquad \text{[Jensen]}$$

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☞ **Optimal value** $r = \sqrt{8/|\widetilde{\delta}|}$.

$$\log \rho(x_0) \leq \log(M|\widetilde{\delta}|/8\pi) + 1 \qquad \Rightarrow \qquad \rho(x_0) \leq (Me/8\pi)|\widetilde{\delta}|.$$

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☞ L^∞ **bound.** $\Delta v \geq \delta \Rightarrow \Delta \log \rho \geq \delta - \rho$

$$\|\rho\|_{L^\infty} \leq \frac{Me}{8\pi} |\delta - \|\rho\|_{L^\infty}| \qquad \Rightarrow \qquad \|\rho\|_{L^\infty} \leq \frac{Me}{8\pi - Me} |\delta|.$$

Subcritical (and critical) mass

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Assume $0 < M < 8\pi$ and $\delta \neq 0$. Let us argue by **contradiction**.

☞ **Sequence**. $\Delta \log \rho_n + \rho_n \geq \delta$ and $\|\rho_n\|_{L^\infty} \geq n\delta$.

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☞ **Scaling**. Consider

$$\eta_n(x) = \lambda_n^2 \rho_n(\lambda_n x) \quad \text{where} \quad \lambda_n^2 = \frac{1}{\|\rho_n\|_{L^\infty}}$$

and we notice

$$\int_{\mathbb{R}^2} \eta_n = M \quad \text{and} \quad \|\eta_n\|_{L^\infty} = 1.$$

Furthermore, $\Delta \log \eta_n + \eta_n = \Delta \log \rho_n(\lambda_n \cdot) + \lambda_n^2 \rho_n(\lambda_n \cdot) \geq \delta \lambda_n^2$.

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Subsolution of the Liouville equation with mass $\leq M < 8\pi$.

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$$\Delta v(t,x) \geq -\frac{C}{t} \quad \text{for a.e. } (t,x) \in (0,T)\times\mathbb{R}^d.$$

$$\|\rho(t,\cdot)\|_{L^\infty} \leq \frac{C}{t} \quad \text{for a.e. } (t,x) \in (0,T)\times\mathbb{R}^d$$

➢ **Subcritical and critical mass**. $\mathcal{F}[\rho_0], \mathrm{m}_2[\rho_0] < +\infty$.
Using Log-HLS and conservation of m_2 we get additional integrability.

$$\Delta v(t,x) \geq -C(T)\left(1+\frac{1}{t}\right) \quad \text{for a.e. } (t,x) \in (0,T)\times\mathbb{R}^d.$$

$$\|\rho(t,\cdot)\|_{L^\infty} \leq C(T)\left(1+\frac{1}{t}\right) \quad \text{for a.e. } (t,x) \in (0,T)\times\mathbb{R}^d$$

Li-Yau estimate

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➢ We construct **solutions** for Keller-Segel.

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Discussion

➢ **Higher dimension**. Aronson-Bénilan to cover higher dimension cases.

➢ **Existence of solutions**. Uniform L^∞ bounds and global in time sol.

➢ **Smoothing of the initial condition**. For small initial mass the solution is immediately smoothed out. Even if $\rho_0 \in \mathcal{M}(\mathbb{R}^d)$

Is this mass condition a **sharp** threshold???

➢ L^∞ **bounds**. New **decays** in time for Keller-Segel.

➢ **Lipschitz estimates for non-local aggregation-diffusion PDEs**.

Thank you for you attention!!!

C. Elbar, A. F-J, and F. Santambrogio. A Li-Yau and Aronson-Bénilan approach for the Keller-Segel with critical exponent. *Preprint in preparation*.

References I

[LY86] P. Li and S. T. Yau. "On the parabolic kernel of the Schrödinger operator". In: *Acta. Math.* 156 (1986), pp. 153–201.

$$\partial_t \Delta v = \Delta(\Delta v) + \nabla(\Delta v) \cdot [2\nabla v + \nabla u] + 2|D^2 v|^2$$
$$+ 2 \left(D^2 v - \frac{1}{2} \Delta v \, \mathrm{Id} \right) : \left(D^2 u - \frac{1}{2} \Delta u \, \mathrm{Id} \right)$$

If $x_0 = x_0(t)$ is a minimum point of $\Delta v(t, \cdot)$

$$\partial_t \Delta v(t, x_0) \geq 2 \left| D^2 v(t, x_0) \right|^2$$
$$+ 2 \left(D^2 v(t, x_0) - \frac{1}{2} \Delta v(t, x_0) \, \mathrm{Id} \right) : \left(D^2 u(t, x_0) - \frac{1}{2} \Delta u(t, x_0) \, \mathrm{Id} \right).$$