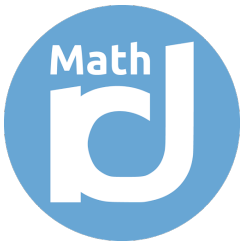
Li-Yau-Hamilton type inequalities for the JKO scheme on the torus

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Everything You Always Wanted to Know About the JKO Scheme





The classical Li-Yau-Hamilton Inequality

Theorem (Li-Yau-Hamilton)

Let $\mathcal{M} = \mathbb{R}^d$ or $\mathbb{T}^d$, ρ positive solution to the heat equation $\partial_t \rho = \Delta \rho$. Then

$$D^2 \log \rho_t \geq -\frac{1}{2t} \tag{1}$$

Generalization:

1. Riemannian manifolds with bounded curvature.
2. Metric spaces, Markov diffusions.
3. Non-Linear equations: p-Laplacian, Porous-Medium.
4. Local-Versions of the estimate.

Extension to periodic Granular-Media equation

What about Fokker-Planck, or Granular-Media equation ?

$$\partial_t \rho = \Delta \rho + \nabla \cdot [\rho(\nabla V + \nabla W * \rho)] \tag{2}$$

Classical maximum principle does not work immediately: there are gradient terms to controls. But on the torus one can uses

Lemma (Semi-Convexity to Lipschitz)

Let $u \in C^{0,1}(\mathbb{T}^d)$, $D^2 u \geq -\lambda$ weakly. Then $|\partial_i u|_\infty \leq \frac{\lambda}{2}$ $\forall i = 1, \ldots, d$.

Extension to periodic Granular-Media equation

Theorem (Li-Yau-Hamilton for periodic Granular-Media)

ρ positive solution to Granular-Media equation on the torus with $V, W \in C^{2,1}(\mathbb{T}^d)$. Then for some $\Lambda > 0$ (explicit)

$$D^2(\log \rho_t + V + W * \rho_t) \geq -\frac{\Lambda}{2} \frac{1}{1 - e^{-\Lambda t}} \tag{3}$$

Some consequences:

1. Quantitative Lipschitz and L^∞ bounds.
2. Quantitative Harnack's inequality for the Granular-Media equation.

Gradient Flow structure of the equation and JKO scheme

Granular-Media equation is a $\mathbb{W}_2$ gradient flow for the energy $\mathcal{F}$:

$$\mathcal{F}[\rho] = \int \rho(\log \rho + V + W * \rho) \tag{4}$$

One can use the JKO scheme to approximate solution

$$\rho^\tau_{k+1} \in \operatorname{argmin}_\rho \mathcal{F}[\rho] + \frac{1}{2\tau} W_2^2(\rho, \rho^\tau_k) \tag{5}$$

piecewise constant interpolation ρ^τ converges weakly (under $\mathcal{F}[\rho_0] < +\infty$) to a weak-solution to the Granular-Media equation.

Li-Yau-Hamilton for the JKO scheme ?

Question: Do we have a Li-Yau-Hamilton type estimate at the level of the JKO scheme ? First answer in the Heat case ($V = W = 0$)

Theorem (P.W. Lee 15')

If $\rho_0 \in C^{2,\alpha}(\mathbb{T}^d)$, $\rho_0 > 0$. *There exists* $C \in (1/2, 1]$ *such that*

$$D^2 \log \rho_t^\tau \succeq -\frac{C}{t} \tag{6}$$

Natural questions

- Can we get rid of the assumptions on ρ_0 ?
- Can we obtain the sharp constant $C = 1/2$?
- Can we extends to V, W non zero.

Li-Yau-Hamilton for the JKO scheme !

Theorem (JKO Li-Yau-Hamilton estimate)

Let $\rho_0 \in \mathcal{P}(\mathbb{T}^d)$. Then

$$D^2[\log \rho_t^\tau + V + W * \rho_t^\tau] \geq -\frac{\Lambda}{2}\frac{1}{1-e^{-\Lambda t}} - \epsilon(\tau) \tag{7}$$

with the $\epsilon(\tau) \to 0$ locally uniformly on $(0, +\infty)$, and independently of ρ_0.

Some consequences:

1. Quantitative (in τ) Lipschitz and L^∞-bounds for ρ_t^τ.
2. Quantitative discrete Harnack inequality.
3. Local in time convergence of the flow in $L_t^2 H_x^2$ and $L_t^\infty C_x^{0,\alpha}$ (based on F. Santambriogio and G. Toshpulatov).

Sketch of proof.

1. Maximum principle à la Caffarelli gives a one-step improvement of semi-convexity along the flow: if one has a lower-bound by λ_k^τ, then we can derive a lower-bound λ_{k+1}^τ satisfying $F[\lambda_{k+1}^\tau, \tau] \leq \lambda_k^\tau$.
2. Discrete comparison principle with the solution to $F[X_{k+1}^\tau, \tau] = X_k^\tau$, $X_0^\tau = +\infty$, gives a uniform bound by a sequence depending only on τ.
3. Careful analysis of the asymptotic for X_k^τ as $\tau \to 0$, $k\tau \sim t > 0$, by comparison with a continuous ODE, gives the result.



Future direction

1. Extension to $\mathbb{R}^d$: ongoing work.
2. Aranson-Bénilan $\Delta \rho_t^{m-1} \geq -\kappa/t$ for solutions to the porous-medium equation: works at the level of the JKO in dimensions 1 and 2, but the method breaks down in larger dimension.
3. Riemannian manifolds ? Other equations ? Local version ?

Thank you for your attention !